THE MATHEMATICAL GAZETTE

EDITED BY
W. J. GREENSTREET, M.A.

WITH THE CO-OPERATION OF
F. S. MACAULAY, M.A., D.Sc.
AND
PROF. E. T. WHITTAKER, M.A., F.R.S.

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A CRITICAL ACCOUNT OF EUCLID'S EXPOSITION OF THE THEORY OF PROPORTION IN THE FIFTH BOOK OF THE "ELEMENTS."

BY PROFESSOR M. J. M. HILL, F.R.S.

(I.)

THIS great work, generally attributed to Eudoxus, has attracted the attention of mathematicians of all time, but is now neglected both in the Schools and in the Universities.

The first person to discover a flaw in the logic of any of the propositions, as given in the text that has come down to us, was (I believe) Saccheri, who, in his *Euclides ab omni naevo vindicatus*, pointed out that in the demonstration of the 18th proposition it was assumed that if there be three magnitudes of which the first and second are of the same kind, then a fourth magnitude of the same kind as the third must exist such that the ratio of the first magnitude to the second was the same as that of the third to the fourth. This proposition he endeavoured unsuccessfully to prove, but it is now known that its validity depends on the Axiom of Continuity.

Then Simson, in his edition of Euclid's Text, noticed that in the demonstration of the 10th proposition Euclid assumes more about ratios than his definitions allow.

Simson, on the principle that the king can do no wrong, ascribes the faulty logic in the proofs of the 10th and 18th propositions to unskilful commentators, who had altered Euclid's original text, and he showed how both these propositions could be rigorously deduced from Euclid's definitions. Simson says, in his notes on the Fifth Book:

"The fifth book being thus corrected, I most readily agree to what the learned Dr. Barrow says:

"'That there is nothing in the whole body of the elements of a more subtile invention, and more accurately handled than the doctrine of proportionals.'

"And there is some ground to hope that geometers will think that this could not have been said with as good reason since Theon's time till the present."

Sir T. L. Heath, quoting the above in his edition of the *Elements*, adds:

"Simson's claim herein will readily be admitted by all readers who are competent to form a judgment upon his criticism and elucidation of Book V."

Further, Sir T. L. Heath, on p. 120 of his article on "Mathematics and Astronomy" in the *Legacy of Greece*, says in regard to the *Elements*:

"No work presumably, except the Bible, has had such a reign; and future generations will come back to it again and again as they tire of the variegated substitutes for it and the confusion resulting from their bewildering multiplicity."

De Morgan, on the first page of his treatise on the *Connexion of Number and Magnitude*, referring to the Fifth Book, says:

"And yet this same book and the logic of Aristotle are the two most unobjectionable and unassailable treatises which ever were written."

(II.)

It is evident therefore that any one who attempts to criticise this great work will be regarded by many persons as trespassing on holy ground with intent to commit sacrilege. And yet there is something which requires explanation. How are we to account for the undoubted fact that the Fifth Book has fallen into disuse, not only in the Schools where its admitted difficulty may be deemed to be a sufficient reason for its omission from the curriculum, but also in the Colleges and Universities where, in view of its inherent importance, that difficulty should not be regarded as an obstacle?

So far as courses of study in Mathematics in the Universities are concerned, the method of the book survives only in its offspring, the definition of the Irrational Number given by Dedekind.

Before going further I desire to recognise the great beauty of the completed structure and to express my admiration for the wonderful genius who erected it. And I also express my agreement in the view that the argument, as corrected by Simson, is logically unchallengeable.

But I hope to prove that for the four reasons enumerated in the last section of this paper the book is not suitable for teaching purposes, and at the same time to show that the subtlety and artificiality of Euclid's argument can without loss of rigour be avoided *by assimilating the proofs of some of the propositions to those given by Euclid himself of some of the others.*

(III.)

In the first place I call attention to the fact that the argument is a complete *tour de force.* Its subtlety is its defect. It is lacking in simplicity, a simplicity which I hope to show is attainable.

Perhaps the strangest feature in the structure of the argument is that it makes no reference to the simplest of all the ideas about ratio, viz.: that idea out of which all other ideas about ratio must have been developed. This idea Euclid states as a proposition in the 5th Proposition of the Tenth Book of the *Elements* in the following terms:

"Commensurable magnitudes have to one another the ratio of a number to a number."

In this statement "number" means "whole number."

Euclid deduces this from the 20th definition of the Seventh Book and the 22nd Proposition of the Fifth Book, so that it depends on the greater part of the argument in the Fifth Book, and therefore is in fact based ultimately on the 5th and 7th definitions of that book and the Axiom of Archimedes.

Now I submit that *the statement set out in the enunciation of the 5th Proposition of the Tenth Book ought not to be regarded as a proposition to be proved, but that it is in fact one of the fundamental definitions or assumptions on which any theory of ratio must be built.*

Thus, if we think of two lengths of 3 feet and 4 feet, we define the ratio of 3 feet to 4 feet as being the same as the ratio of 3 to 4, and we write the value of this ratio as the fraction 3/4, which we take to be the equivalent of Euclid's 3 : 4.

There are two other fundamental assumptions to be made, which are most easily reached by considering special cases of the above definition:

(i) Suppose there are two lengths of 3 feet and one length of 4 feet. We say that the ratio of each length of 3 feet to the length of 4 feet is the same as that of 3 to 4, and that therefore they are equal.

By generalising this example we obtain the idea that if there are three commensurable magnitudes A, B and C; and if A is equal to B, then $(A:C)=(B:C)$; and we readily proceed to the further generalisation that the same holds good when A and B are not commensurable with C.

This then is a fundamental assumption, but Euclid deduces it in the first part of V. 7 as a consequence of his definitions.

(ii) Suppose that there are three lengths of 7, 6 and 5 feet respectively. We admit at once that the ratio of 7 feet to 5 feet is greater than the ratio of 6 feet to 5 feet.

Generalising this in like manner we obtain the idea that if there are three commensurable magnitudes, A, B and C, and if A be greater than B, then the ratio of A to C is greater than that of B to C.

And we then extend this to the case in which C is not commensurable with A and B.

This again is a fundamental assumption, but Euclid deduces it in the first part of V. 8 as a consequence of his definitions.

These three assumptions, together with the Axiom of Archimedes, seem to me to form the bed-rock on which any theory of ratio must be built.

(IV.)

But the argument in the Fifth Book proceeds very differently. It is solidly based on three definitions.*

These are (i) The fourth definition. *Magnitudes are said to have a ratio to one another which are capable when multiplied of exceeding one another.*

This quite clearly involves the so-called Axiom of Archimedes, viz.: if A and B are two magnitudes of the same kind, then a whole number n exists, such that nA is greater than B.

This Axiom is only a refinement of the Arabian proverb that "It is the last straw that breaks the camel's back." And it is a misnomer to connect it with the name of Archimedes, but I conform to the usual practice lest I should be misunderstood.

(ii) The fifth definition.

This is equivalent to the following:

$$\text{If } (A:B)=(C:D),$$

then if r, s are *any* positive integers *whatever*,

and	if $rA > sB$,	then must also $rC > sD$; ....(I)
but	if $rA = sB$,	then must also $rC = sD$; ....(II)
and	if $rA < sB$,	then must also $rC < sD$. ....(III)

Euclid gives no explanation of the way in which he obtained this definition, and it is therefore a cause of immense difficulty to the beginner.

Moreover, it has been proved, I believe first, by Stolz in his *Vorlesungen über allgemeine Arithmetik*, Theil I, Seite 87 (1885), that if Archimedes' Axiom hold, then the set of conditions marked (II) above is superfluous.

If the set of conditions marked (II) be omitted I will call what remains the *simplified* form of the 5th definition.

* I leave out of account Euclid's definition of ratio, because he makes no use of it in his argument.

We may express this definition in the following equivalent form:

" *Two ratios are equal when* no *rational fraction* whatever *lies between them.*"

(iii) The seventh definition.

This is equivalent to the following:

$$(A : B) > (C : D)$$

if any *single* pair of integers r, s exist such that $rA > sB$, but either $rC < sD$ or $rC = sD$. Euclid does not give any explanation of the way in which he obtained this definition.

Moreover, it can be shown by the aid of Archimedes' Axiom that it is unnecessary to consider the case in which $rA > sB$, $rC = sD$; because it can be proved * that then other integers r', s' exist such that $r'A > s'B$, $r'C < s'D$.

So it is sufficient to say that $(A : B) > (C : D)$ if any *single* pair of integers r, s exist such that

$$rA > sB, \quad rC < sD.$$

We may express this definition in a form in which it is at once seen that it is the converse of the fifth definition.

" *Two ratios are unequal if* some *rational fraction lies between them.*"

It seems to me to be evident, on the face of it, that the 5th and 7th definitions were not the original definitions from which Euc. X. 5, Euc. V. 7 (Part 1) and Euc. V. 8 (Part 1) were first obtained, but that on the contrary Euc. V. def. 5 and Euc. V. def. 7 were deduced from the ideas involved in the enunciations of Euc. X. 5, V. 7 (Part 1) and V. 8 (Part 1), and the Axiom of Archimedes.

I suggest that this was effected by some means equivalent to the following:

Suppose A and B are magnitudes of the same kind, and that the multiples of A are compared with those of B.

Suppose it is found that $rA > nB$.

Divide B into r equal parts.† Call each part G.

$$\therefore\ B = rG;$$
$$\therefore\ rA > n(rG);$$
$$\therefore\ rA > r(nG);$$
$$\therefore\ A > nG.$$

$\therefore\ (A : B) > (nG : B)$ by the idea in Euc. V. 8 (Part 1);

$$\therefore\ (A : B) > (nG : rG);$$

but $(nG : rG) = (n : r)$ by the idea in Euc. X. 5;

$$\therefore\ (A : B) > (n : r).$$

And now we express $(n : r)$ in the form n/r;

$$\therefore\ (A : B) > n/r.$$

So that, if $rA > nB$, then $(A : B) > n/r$.

Similarly, if $rA = nB$, then $(A : B) = n/r$;

and if $rA < nB$, then $(A : B) < n/r$.

From these three results, by means of a *reductio ad absurdum*, the converse proposition follows:

If $(A : B) > n/r$, then $rA > nB$;

if $(A : B) = n/r$, then $rA = nB$;

and if $(A : B) < n/r$, then $rA < nB$.

Having thus shown how to compare the ratio of two magnitudes of the same kind with any rational fraction, the next ideas that arise are

(i) *Two ratios are equal when* no *rational fraction* whatever *lies between them.*

---

* See the writer's *Theory of Proportion*, p. 88. (Constable & Co., 1914.)

† Euclid assumes the possibility of such division in X. 6.

From this Euc. V. def. 5 can be deduced in Euclid's form. This definition is the test for equality of ratios. It should be noticed that in constructing it the fundamental assumption regarding unequal ratios (viz.: If $A > B$, then $(A : C) > (B : C)$) was employed.

(ii) *Two ratios are not equal if* some *rational fraction lies between them.*

From this Euc. V. def. 7 can be deduced in a form which is equivalent to that given by Euclid.

(V.)

Having now felt our way from the fundamental ideas about ratio to the basic definitions of Euclid's Fifth Book, I proceed to classify its contents.

(1) There are a group of propositions about magnitudes and their *multiples* (not their *ratios*).

These are the 1st, 2nd, 3rd, 5th and 6th propositions. Their proofs do not depend on Archimedes' Axiom. With them it is convenient to associate a proposition, which, though not separately stated, is involved in the first part of the 8th proposition. It does not involve ratios but does depend on Archimedes' Axiom.

It is as follows:

If A, B, C are magnitudes of the same kind, and if A be greater than B, then whole numbers n and t exist such that $nA > tC > nB$.

Since A is greater than B, therefore $A - B$ is a magnitude of the same kind as C, and therefore by Archimedes' Axiom a whole number n exists such that

$$n(A - B) > C;$$

$$\therefore\ nA > nB + C,\text{*}$$

so that nA exceeds nB by more than C.

By Archimedes' Axiom a multiple of C exists which exceeds nB.

Let tC be the smallest multiple of C which exceeds nB.

Then tC does not exceed nB by more than C, but nA exceeds nB by more than C.

$$\therefore\ nA > tC \quad \text{and} \quad tC > nB;$$

$$\therefore\ nA > tC > nB.$$

I shall have to refer to this proposition later, and for the purpose of this paper I will call it *the Subsidiary Proposition.*

Euclid's proof of it occupies the greater part of the proof of Prop. 8, and is very complex and cumbrous.

(2) There is a group of propositions concerned with *unequal* ratios.

These are the 8th, 10th and 13th propositions.

The only use made of them by Euclid is to prove properties of equal ratios in the Fifth Book. They are never used in any other part of Euclid's Elements.

Now, if the test for the Equality of Ratios (Euc. V. Def. 5) *is a sound and complete one, it should suffice to prove all the properties of equal ratios, and it ought not to be necessary to bring in the properties of unequal ratios to prove properties of equal ratios.*

This is in fact the case.†

What I have called the Subsidiary Proposition contains all that is effective in Euc. V. 8, 10 and 13, when these propositions are employed to prove properties of equal ratios.

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* The rest of the work amounts to proving that if $X > Y + Z$, then an integer t exists such that $X > tZ > Y$.

† As has been previously remarked, the fundamental assumption regarding unequal ratios was employed in constructing the test for equality of ratios, but the test for equality of ratios having once been found, propositions concerned with unequal ratios need not be employed to prove propositions about equal ratios.

(3) The next group of propositions consists of the following:

Euc. V. 4. If $(A:B)=(C:D)$, then $(rA:sB)=(rC:sD)$.

The so-called Corollary to Euc. V. 4.

$$\text{If } (A:B)=(C:D), \quad \text{then } (B:A)=(D:C).$$

V. 7. If $A=B$, then $(A:C)=(B:C)$ and $(C:A)=(C:B)$.

V. 11. If $(A:B)=(C:D)$ and if $(C:D)=(E:F)$, then $(A:B)=(E:F)$.

V. 12. If $(A:B)=(C:D)=(E:F)$, then $(A+C+E:B+D+F)=(A:B)$.

V. 15. $(A:B)=(nA:nB)$.

V. 17. If $(A:B)=(C:D)$, then $(A-B:B)=(C-D:D)$.

V. 18. If $(A:B)=(C:D)$, then $(A+B:B)=(C+D:D)$.

Euclid gives proofs of these (excepting Prop. 18) which depend only on the 5th definition and the propositions concerning the multiples of magnitudes in the first group, excluding the Subsidiary Proposition, and Simson gave such a proof of Prop. 18.*

The proofs of all these propositions can be arranged so as to be independent of one another. They depend only on the 5th definition and propositions concerning multiples of magnitudes, excluding the Subsidiary Proposition. Such proofs should (I think) be regarded as belonging to the *normal* type. They can all be made to proceed in the following way:

In each proposition it is required to prove that some two ratios are equal. One of these ratios is compared with *any* rational fraction *whatever*, say r/s. There are then three possibilities.

The selected ratio may be (i) greater than, or (ii) equal to, or (iii) less than r/s. It is then proved by the aid of the data that the other of the two ratios is in the first case greater than, in the second case equal to, and in the third case less than r/s.

Hence r/s cannot lie between the two ratios.

But r/s represents *any* rational fraction *whatever*.

Therefore *no* rational fraction *whatever* can lie between the two ratios. Therefore the two ratios are equal.

The steps of the argument in each proposition follow one another inevitably and automatically. This is true even in regard to the 18th proposition. There is no other property of equal ratios to be used in the proof of any one of the propositions, and consequently none to be sought for. It is the necessity of making this search at several stages of Euclid's proofs of such propositions as Euc. V. 16, 22, 23 and 24 that makes them so difficult, not indeed to follow when the road is pointed out, but to master and remember.

Each of the proofs of what I have called the *normal* type is a good example of Dedekind's theory of the "cut" in the system of rational numbers.

(4) The next group consists of:

Euc. V. 16. If $(A:B)=(C:D)$, and if all the magnitudes are of the same kind, then $(A:C)=(B:D)$.

Euc. V. 22. If $(A:B)=(T:U)$ and if $(B:C)=(U:V)$, then $(A:C)=(T:V)$

Euc. V. 23. If $(A:B)=(U:V)$ and if $(B:C)=(T:U)$, then $(A:C)=(T:V)$.

Euc. V. 24. If $(A:C)=(X:Z)$ and if $(B:C)=(Y:Z)$, then

$$(A+B:C)=(X+Y:Z).$$

The proofs of the first three of these differ from those of the normal type only in that they require the use of Archimedes' Axiom at an early stage, *the Subsidiary Proposition* and the *simplified* form of the 5th definition.

In other respects the proofs of Euc. V. 16, 22 and 23 follow the *normal* type very closely, and are very similar to each other. Each proof is independent

* Simson's proof can be put much more simply (see the writer's *Contents of Euc. V. and VI.* 2nd edition, p. 113).

of all other properties of equal ratios, and each is a good illustration of Dedekind's theory of the "cut" in the system of rational numbers. Every step seems to follow inevitably from the preceding. These proofs will be found in the second edition of the writer's *Contents of the Fifth and Sixth Books of Euclid.*

The proof of the 24th proposition is more difficult than those of the other three, although there is a general resemblance to the proofs of those propositions. It is given in the writer's "Fifth Paper on the Fifth Book of Euclid's *Elements*" published in the *Transactions of the Cambridge Philosophical Transactions* for 1921. It is also a beautiful illustration of Dedekind's theory of the "cut" in the system of rational numbers. (The proofs of the Corollaries of Euc. V. 24 can be worked out on the same lines, but they are more difficult.) On the other hand, Euclid's proofs of the propositions in this group depend in each case on other properties of equal ratios. It is not obvious at each stage which properties are to be made use of. This constitutes a great strain on the memory, though it may be freely confessed that it increases our admiration for the genius who first obtained these proofs. The proofs in Euclid's form do not furnish direct illustrations of Dedekind's theory of the "cut" in the system of rational numbers. They are rather examples of extremely ingenious applications of other properties of equal and unequal ratios previously proved.

(5) The next group of propositions contains only the single proposition 19.* I have not up to the present obtained a simple and direct proof of it conforming to the *normal* type.

If, however, it is pointed out that it is merely a transformation of the hypothesis and conclusion of Proposition 17 by the aid of Proposition 16, it cannot constitute a difficulty to the student.

(6) The last group of propositions differs in kind from the 3rd, 4th and 5th groups. It is not therefore to be expected that their proofs could or should conform to the *normal* type. The object in these propositions is not to demonstrate the equality of two ratios, but to prove that certain magnitudes occurring in the ratios are equal or unequal as the case may be.

These are:

V. 9. If $(A : C) = (B : C)$ or if $(C : A) = (C : B)$, then $A = B$.

V. 14. If $(A : B) = (C : D)$ and if the magnitudes are all of the same kind, then $A \gtreqless C$ according as $B \gtreqless D$.

V. 20. If $(A : B) = (T : U)$ and if $(B : C) = (U : V)$, then $A \gtreqless C$ according as $T \gtreqless V$.

V. 21. If $(A : B) = (U : V)$ and if $(B : C) = (T : U)$, then $A \gtreqless C$ according as $T \gtreqless V$.

V. 25. If $(A : B) = (C : D)$ and if A be the greatest of the four magnitudes A, B, C, D, then $(A + D) > (B + C)$.

Euclid's proof of V. 9 is an immediate application of what has been called the Subsidiary Proposition.

It is here suggested that Euc. V. 16 should be proved independently of all other properties of equal ratios directly from the 5th definition, and then Prop. 14 becomes unnecessary, or can be immediately obtained from Prop. 16.

The same kind of relation as that between Props. 14 and 16 exists between Props. 20 and 22 and also between Props. 21 and 23.

Thus Props. 14, 20 and 21 are unnecessary if Props. 16, 22 and 23 are independently proved.

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* If $A : B = C : D$, then $A - C : B - D = A : B$.

Prop. 25 is obtained by Euclid from Props. 7, 11, 14 and 19. It can be obtained also from Props. 16 and 17. It is, however, never used by Euclid in any other part of the *Elements*, and I believe once only in Greek Geometry (see the 69th Theorem of the 7th Book of the *Collectiones Mathematicae* of Pappus).*

The 25th proposition includes as a particular case the following: *The arithmetic mean of two magnitudes is greater than their geometric mean.* It can also be made the basis of the determination of two important limits:

$$\text{(i)}\ \underset{n\to+\infty}{L}\ a^n=+\infty \quad \text{if } a>1.$$

$$\text{(ii)}\ \underset{n\to+\infty}{L}\ a^n=+0 \quad \text{if } 0<a<1.$$

(See the 2nd article by the writer on the "Fifth Book of Euclid's *Elements*," *Cambridge Philosophical Transactions*, vol. xix. pp. 170-172.)

Thus, though Euclid makes no use of it, it is a very suggestive proposition.

(VI.)

The chief obstacles encountered by the student of Euclid's Fifth Book are:

(1) The inversion of the natural order of ideas.

(2) The omission of all explanation of the manner in which the 5th and 7th definitions were originally obtained.

(3) The unnecessary use of properties of *unequal* ratios to prove properties of *equal* ratios, and the consequent indirectness and artificiality of the proofs.

(4) The absence of any uniform plan upon which Euclid's proofs are based.

For these reasons I think that the use of Euclid's text of the Fifth Book will never be resumed in the Universities in the future. On the other hand, I believe that a statement of the argument on the lines indicated in the above would be of great value to University students commencing a serious study of the Calculus, to whom a knowledge of Dedekind's theory of the "cut" in the system of rational numbers is essential. I have found no difficulty in teaching the subject to such students for many years past. I do not advocate the introduction of the subject into the ordinary school curriculum.

For ordinary school use the idea of the ratio of commensurable magnitudes might very well be introduced into the teaching of Elementary Geometry immediately after the pupil has learned that parallelograms on equal bases and between the same parallels are equal, and again after he has learned that in equal circles equal angles at the centre subtend equal arcs; the theory of similar figures might be limited to the discussion of cases in which the magnitudes concerned are commensurable. The beginner might be shown how to divide a triangle into n^2 congruent triangles each similar to the whole. This would form the basis of the proof of the proposition that the areas of similar triangles are proportional to the squares constructed on corresponding sides.

M. J. M. Hill.

GLEANINGS FAR AND NEAR.

150. Notwithstanding the foregoing injunctions of Dr. Cornelius, he yet condescended to allow the child the use of some few modern playthings; such as might prove of any benefit to his mind, by instilling an early notion of the sciences. For example, he found that marbles taught him percussion and the laws of motion; nutcrackers the use of the lever; swinging on the ends of a board the balance; bottle-screws the vice; whirligigs the axis in peritrochia; bird-cages the pulley; and tops the centrifugal motion.—*Martinus Scriblerus*, c. v.

* I am indebted to Sir T. L. Heath for this reference.

PURE MATHEMATICS IN A SECONDARY SCHOOL ADVANCED COURSE.

By F. G. Hall, M.A.

Much has been said about the danger of dividing the various branches of pure mathematics into "water-tight compartments," and books have been written with a view to breaking down from the very beginning the barriers between the different subjects.

I feel, however, that to the average boy who has not yet reached Matriculation standard, the division of Mathematics into Arithmetic, Algebra and Geometry is very real and most helpful. A straight-forward question in mensuration, the study of the identity $(a+b)^2=a^2+2ab+b^2$, and the theorem that the square on a line is equal to the sum of the squares on its two segments together with twice the rectangle contained by them, for example, will and should appear as separate questions—the first being linked with other similar exercises in Arithmetic, the second depending on algebraical multiplication and being a study in form rather than in area, and the third turning on a geometrical construction. Only after each has been treated in the appropriate way should we show the pupil the cross-connections existing in the three problems.

We must, of course, encourage the application of knowledge gained in one subject to the solution of difficulties in another—the processes of Algebra will be taught by regarding them as extensions of the corresponding work in Arithmetic, and harder problems in Arithmetic and Geometry will be attacked by the methods and symbolism of Algebra—but I believe that each branch must be carried independently up to a certain point before all can profitably be developed together.

This point would seem to be reached where the Advanced Course begins.

If, after this, Algebra, Geometry, Trigonometry and the Calculus are treated as separate subjects, much time will be wasted and the unity of mathematical knowledge will not be grasped. The same kind of work will be repeated at different times. We have suffered much in the past because ratio and proportion have been treated algebraically, geometrically and trigonometrically; logarithms have occurred in every type of mathematical text-book; and variation has been considered—at odd intervals and without any correlation—from the points of view of formal algebra, algebraic graphs, trigonometrical graphs and the Calculus. Not only does this involve a huge waste of time and tend to obscure the unity of the whole subject, but it also leads to very difficult work being attempted in one subject before the much easier beginnings are made in another.

This article is an attempt to outline a continuous course in Pure Mathematics for the two years after Matriculation—a course in which there shall be as little repetition as possible, in which the easy parts of the different branches shall come first and together, and in which a much wider range of subjects will be covered than can at present be attempted.

I have supposed it possible to divide into seven chapters all the work that should be done, and have outlined what would seem to be the best sequence for the chapters and the probable contents of each.

I feel that the publication of these views may lead to very healthy criticisms of them and to the elaboration of a better scheme on which an adequate treatment of the subject may ultimately be built.

Chapter I.—Ratio in Algebra, Geometry and Trigonometry.

A fuller treatment of this important subject and a closer correlation of the work than is usually given would be attempted in this chapter.

The fundamental idea would be stated somewhat as follows: "Given any two quantities, if a unit can be found which is contained exactly in both, then the number of times it is contained in the first divided by the number of times it is contained in the second is called the ratio of the first quantity to the second. If such a unit can be found the quantities are said to be commensurable." Particular examples would be taken from pure numbers, lengths, areas and volumes to illustrate when this is possible and when impossible.

The work in Algebra should be limited to that needed to ensure accuracy and facility in the use of ratios and to the establishment and use of those equalities of ratios which prove helpful in higher work.

In Geometry the usual work on the proportional division of straight lines (theoretical and practical) leading to the different conditions under which triangles are similar would be taken.

This would be followed at once by the elements of Trigonometry up to the solution of right-angled triangles, involving the use of trigonometrical, but not of logarithmic, tables. Much practice in the uses of the trigonometrical ratios would be given in view of the work to be done in the next chapter.

Chapter II.—The Inter-relation of Trigonometry and Geometry.

It is very important that a pupil should realise as soon as possible how powerful are the tools which trigonometry can supply for the solution of geometrical problems, and how trigonometry is developed by the aid of geometrical constructions.

Examples of the application of trigonometry to plane geometry would be given by considering the properties of the triangle and its circles—work which should be taken at a much earlier stage than is customary—the connections between the regular polygons and their circles, and those relations between similar irregular polygons which become as easy to prove when use is made of the trigonometrical expression for the area of a triangle. The first of these would be developed so as to include most of the relations between the sides and angles of a triangle, but the adaptation of the resulting formulae to the use of logarithms would be left until the subject is again taken up in Chapter V.

The second section of Chapter II. would contain only the really important theorems out of those generally included under the heading: "Solid Geometry, Lines and Planes." Much more numerical and graphical work than is usual would be given in connection with these, and trigonometry would be used wherever possible. The whole subject lends itself admirably to trigonometrical treatment.

The last section would contain geometrical proofs of the expansions for $\sin(A+B)$ and $\cos(A+B)$. From these would be deduced the corresponding values of $\sin(A-B)$, $\cos(A-B)$ and $\tan(A\pm B)$. These are needed for the work outlined in Chapters III. and IV., and the subject should be developed up to the point from which the further work in analysis discussed in Chapter VII. can start.

Chapter III.—Variation.

The keynote of this chapter would be what has been called "the idea of functionality."

Linear, quadratic and cubic expressions would be graphed, accurately at first and then more quickly so as to indicate only the salient features of the graphs. Direct variation and variation as the square or the cube of the unknown would be discussed as special cases. The graphs of $\frac{a}{x}$, $\frac{a}{x^2}$ and $\frac{a+bx}{c+dx}$ would also be studied, inverse variation being treated as only one particular case out of many.

Some of the more important trigonometrical graphs would be considered—perhaps as indicated in my article on the subject in the *Gazette* for May 1922.

With all this graphical work would be linked the approximate solution of algebraic and trigonometrical equations and a certain amount of the theory of equations.

The work indicated above would lead to further work in co-ordinate geometry, and this would be developed to include at least the simplest forms of the equations for the circle, parabola, ellipse and hyperbola.

Finally there would be a section dealing in a simple manner with continuity, limits and gradients, explaining the notation of the Differential Calculus and preparing the way for the work of Chapter IV.

Chapter IV.—The Elements of the Differential Calculus and its Application to Algebra, Geometry and Trigonometry.

The introduction of this subject at such an early stage of the Advanced Course is justifiable mainly on two scores: in the first place, a fairly large amount of time is available for thoroughly driving home the fundamental ideas by means of graphical and numerical work—really necessary work, which, however, proves tedious to older pupils; and secondly, only a very few results need be proved before the use and importance of the Calculus is obvious to the pupil—the more generalised and difficult parts of the subject can be postponed to a later stage.

These two principles of putting clearly the chief ideas of the Calculus and of using its results as soon as possible should solve the question of what must be included in this chapter. The subject matter may be briefly outlined as follows: the idea of a differential coefficient as the limit of the ratio $\frac{\delta y}{\delta x}$ when both quantities tend to zero, with illustrations from geometry and mechanics; finding the differential coefficients of powers of x from x^{-3} to x^5 and of $\sin x$, $\cos x$, $\tan x$; the study of increasing and decreasing functions, algebraic and trigonometric, with maxima and minima as special cases; really important problems in maxima and minima from algebra, geometry and trigonometry; the use of the Calculus in plotting algebraic and trigonometric graphs, with some discussion of the significance of the second differential coefficient; the equations of tangents to the various curves studied in Chapter III.; and finally any general results in the Calculus which will be raised by the above work.

The inverse operation of "guessing" $f(x)$ when $f'(x)$ is known would be lightly treated.

Chapter V.—Logarithms and their Use in Arithmetic, Algebra and Trigonometry.

This chapter is intentionally inserted between the last chapter on the Differential Calculus and Chapter VI. on the Integral Calculus, with a view to giving the pupil time to digest the ideas of the former before he tries to assimilate a second set of new ideas. The subject matter of the chapter is of course very important in itself, and would have to be known in any case before any considerable application of the Integral Calculus to the mensuration of areas and volumes could be made.

The usual elementary treatment of the subject—definition; rules for the logarithm of a product, a quotient, a power and a root; logarithms and antilogarithms with 10 as base—would be followed in the first section, but the work would be applied to problems in Algebra and Trigonometry as well as to those of Arithmetic.

Much more emphasis than usual, however, would be laid on the proper adaptation of formulae to the use of logarithms. Practice would be given in

the logarithmic solution of right-angled triangles where the numbers involved are not simple enough to justify the direct use of Pythagoras' Theorem; and the formulae found in Chapter II. for the solution of any triangle would now be transformed into the corresponding "logarithmic" formulae. Compound Interest would also be discussed at this stage.

A short section would deal with the accuracy obtainable with four-figure and with five-figure logarithms in general and in special cases such as "Find the value of $39\frac{1}{2} - 4\pi^2$, and state the probable percentage error in your result."

The evaluation of particular logarithms to base 10 must of course be postponed to Chapter VII., but time can be very profitably spent in finding approximate values of the logarithms of numbers between 1 and 100 from a carefully drawn graph of $y = 10^x$. This will afford opportunity for revision and extension of the "laws of indices."

Chapter VI.—The Elements of the Integral Calculus and its Application to Mensuration.

If Integration is first treated as the reverse process to Differentiation—and this seems to be the method of most text-books—the following difficulties arise:

(1) the meaning of the symbol $\int ... \, dx$ is never perfectly clear, especially if the use of the separate symbol dx has been forbidden in the previous work.

(2) There is always some confusion later between indefinite and definite integrals.

(3) The real significance of integration, as a method of summation, is not grasped, and its application to mensuration may become a mechanical process with the vital point obscured.

Hence in this chapter integration as summation would be the keynote, and the connection between integration and differentiation would *follow* the process of finding certain integrals.

Some preliminary work would be needed on the summation of the series Σn, Σn^2, Σn^3 and $\Sigma \sin nx$, but this only means that these series would appear in their proper setting instead of occurring as "Miscellaneous Series" in Algebra and Trigonometry.

Operations like $\int x^2 dx$, which really means $\int_0^x x^2 dx$, would be defined as methods of finding the limits of series like

$$(\delta x)^2 . \delta x + (2\delta x)^2 \delta x + (3\delta x)^2 \delta x + ... + (n\delta x)^2 \delta x,$$

when $n \to \infty$ and $\delta x \to 0$, where $n\delta x = x$. The corresponding geometrical interpretation would of course be given. Similarly, operations such as $\int_a^b x^3 dx$ would be defined as summations from $x = a$ to $x = b$, and hence as equivalent to

$$\int_0^{x=b} x^3 dx - \int_0^{x=a} x^3 dx.$$

In all cases the sign $\int$ merely denotes summation, whilst dx has its usual double significance, *i.e.* that x is the variable and that a limit is being sought.

Thus, by actual summation, the values of $\int x\,dx$, $\int x^2 dx$, $\int x^3 dx$, $\int \sin x\,dx$, and of the corresponding definite integrals, would be found.

It would then be noticed that the results obtained are strikingly similar to the results of Chapter IV., and the connection between integration and differentiation would be sought—leading to the result that integration and differentiation are practically inverse operations and necessitating a discussion

of the "constant of integration." The other integrals needed for the second half of this chapter would then be found from the corresponding results of Chapter IV.

The application of this work to mensuration would include: the areas of circles, ellipses and segments of these and of parabolas; and the volumes of portions of cones, spheres and other important solids—the whole being treated as suggestively as possible. The aim would be to help a pupil to *use* the powerful methods of the Integral Calculus: so that if he has to find, for example, the volume of a frustum of a given sphere he will *not* quote the formula, volume $=\frac{\pi k}{6}(3r_1^2+3r_2^2+k^2)$, but will attack the problem *ab initio* as a question in summation for which the Integral Calculus provides a short method of solution.

Chapter VII.—Elementary Analysis.

The first part of this chapter would be devoted to the important expansions of Algebra—the Binomial, Exponential and Logarithmic Theorems. The first of these would be proved for a positive integral index only, enough work in Permutations and Combinations being done to enable the usual proof to be thoroughly understood; but the *use* of the generalised theorem and the condition under which the expansion is valid would be indicated. This would be followed by a short section dealing with the convergence of infinite series, well illustrated by examples. After this, the best of the usual "proofs" of the Exponential and Logarithmic Theorems would be studied, and any weaknesses would at least be discussed even though at this stage they cannot be remedied. Exercises on the uses of these two theorems would complete the work in Algebra.

The work done in Trigonometry would now be further developed—perhaps as outlined in my article in the *Mathematical Gazette* for October 1922, so far as the study of "form" is concerned, with additional work on De Moivre's Theorem and the expansions to which it leads.

The last part of the chapter would deal with the more general results of the Calculus dependent on the above work and some easy treatment of the following: Rolle's Theorem and the First Mean-Value Theorem; the different substitutions employed to effect the important types of integration; successive differentiation; and elementary differential equations.

The underlying ideas of the chapter would be (i) the analytical study of "form" in Pure Mathematics, and (ii) the development of manipulative power to enable the study of Higher Mathematics to be undertaken when the pupil proceeds to a University Course.

F. G. Hall.

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151. Louis Philippe, wandering in Switzerland, was made *professeur* in the College of Reichenau, at the age of 22, under the name of Chabot, Oct. 1793. His salary was 1400 frs. per month, and he stayed for 15 months, conducting himself with scrupulous regularity. He taught mathematics, geography, history, French and English. He was unaffected and simple, and no suspicion was aroused as to his rank.

152.
Talia monstrantem justis celebrate camoenis,
Vos qui caelesti gaudetis nectare vesci,
Newtonum clausi reserantem scrinia veri,
Newtonum Musis carum, cui pectore puro
Phoebus adest, totoque incessit numine mentem:
Nec fas est propius mortali attingere Divos.—Halley.

153. Sir William Jones to the famous Doctor Samuel Parr on his vile handwriting: "Your Greek is wholly illegible—it is perfect algebra."

CONJUGATE DIAMETERS IN AREALS.

By N. M. Gibbins, M.A.

Principles. 1. If K be the centre of a conic and P a point not on it, the diameter conjugate to KP is parallel to the polar of P.

2. Conjugate diameters form a pencil in involution. Hence if $S=0$ and $S'=0$ be the equations of two known pairs of conjugate diameters, the general equation is $S+\lambda S'=0$.

The method then is to find the diameters conjugate to KA, KB, KC, where ABC is the triangle of reference, by means of 1, and then to write down the general equation by means of 2.

Ex. 1. Conic touching the sides of the triangle of reference at the middle points.

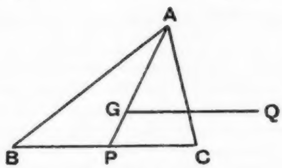

Fig. 1.

The centre of the conic is at $G(1, 1, 1)$ and the equation of GP is $y-z=0$ or $(z-x)+(x-y)=0$.

The equation of GQ, parallel to BC, is $3x=x+y+z$ or $(z-x)-(x-y)=0$.

The combined equation of GP and GQ is therefore $(z-x)^2-(x-y)^2=0$, and the general equation of conjugate diameters is

$$\Sigma\lambda\{(z-x)^2-(x-y)^2\}=0 \quad \text{or} \quad \Sigma(\mu-\nu)(y-z)^2=0.$$

Ex. 2. Conic referred to self-polar triangle.

Equation is $\Sigma ux^2=0$, and centre (x', y', z') is given by $ux'=vy'=wz'$.

AK is $yz'-y'z=0$, and line through K parallel to BC, or $x=0$, is $x\Sigma x'=x'\Sigma x$, *i.e.* $(zx'-z'x)-(xy'-x'y)=0$.

Combined equation is $(yz'-y'z)\{(zx'-z'x)-(xy'-x'y)\}=0$.

Writing down the other two, multiplying in order by λ, μ, ν and adding, the general equation is

$$\Sigma\frac{\mu-\nu}{yz'-y'z}=0 \quad \text{or} \quad \Sigma\frac{\mu-\nu}{u(vy-wz)}=0.$$

Ex. 3. Circum-conic with given centre (x', y', z').

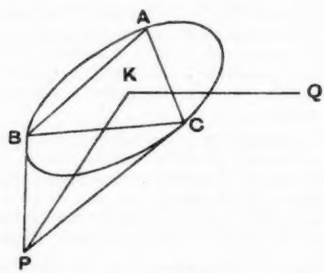

Fig. 2.

KQ as before is $(zx'-z'x)-(xy'-x'y)=0$, and KP is the line joining (x', y', z') to the middle point of BC, *i.e.* to $(0, \frac{1}{2}, \frac{1}{2})$;

$$\therefore \text{ its equation is } \begin{vmatrix} 0, & 1, & 1 \\ x', & y', & z' \\ x, & y, & z \end{vmatrix} = 0 \quad \text{or} \quad (zx'-z'x)+(xy'-x'y)=0.$$

Combined equation is $(zx'-z'x)^2-(xy'-x'y)^2=0$, and general equation is $\Sigma(\mu-\nu)(yz'-y'z)^2=0$.

Ex. 4. General conic $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$.

With the usual notation the centre K is given by

$$\frac{X'}{1}=\frac{Y'}{1}=\frac{Z'}{1}=\frac{xX'+yY'+zZ'}{x+y+z}=\frac{\Delta x'}{A+H+G}=\frac{\Delta y'}{H+B+F}$$

$$=\frac{\Delta z'}{G+F+C}=\frac{\Delta(x'+y'+z')}{A+B+C+2F+2G+2H}\equiv\rho(x'+y'+z'). \quad \text{.........}(1)$$

Equation of AK is $yz'-y'z=0$ and the polar of A is $X=0$.

Line through K parallel to this is $X\Sigma x'=X'\Sigma x$ or $\Sigma ax\,\Sigma x'=\Sigma xX'$ by equations (1).

Compare this with $\quad \beta(zx'-z'x)-\gamma(xy'-x'y)=0, \quad$ ....(2)

i.e. with $\quad x(\beta z'+\gamma y')-\gamma x'y-\beta x'z=0$;

$$\therefore \frac{\beta}{g(x'+y'+z')-Z'}=\frac{\gamma}{h(x'+y'+z')-Y'} \quad \text{or} \quad \frac{\beta}{g-\rho}=\frac{\gamma}{h-\rho}, \text{ by above.}$$

The coefficient of x is consistent with these ratios. Hence (2) becomes

$$(g-\rho)(zx'-z'x)-(h-\rho)(xy'-x'y)=0.$$

Combining with $(f-\rho)(yz'-y'z)$, and proceeding as before, the general equation of conjugate diameters is

$$\Sigma\frac{\mu-\nu}{f-\rho}\cdot\frac{1}{yz'-y'z}=0,$$

where x', y', z' and ρ are given by equations (1).

Equation of the Axes of the Conic.

The equation of the conjugate diameters may be written

$$\Sigma p(zx'-z'x)(xy'-x'y)=0, \quad \text{.........}(1)$$

where

$$\Sigma(f-\rho)p=0. \quad \text{.........}(2)$$

In (1) coefficient of x^2 is $-py'z'$ and that of yz is $x'(-px'+qy'+rz')$. The axes are perpendicular conjugate diameters, and the condition is

$$\Sigma a^2py'z'+\Sigma bc\cos Ax'(-px'+qy'+rz')=0. \quad \text{.........}(3)$$

In this the coefficient of p is easily found to be

$$a^2y'z'+x'(-bc\cos Ax'+ca\cos By'+ab\cos Cz'),$$

which

$$=a^2y'z'+b^2z'x'+c^2x'y'-x'\{bc\cos Ax'+c(c-a\cos B)y'+b(c-a\cos C)z'\}$$

or

$$\Sigma a^2y'z'-x'bc\cos A\Sigma x', \quad \text{.........}(4)$$

and therefore condition (3) becomes

$$\Sigma a^2y'z'\,\Sigma p-\Sigma bcpx'\cos A\,\Sigma x'=0. \quad \text{.........}(5)$$

Eliminating p, q, r between equations (1), (2) and (5), the equation of the axes may be written

$$\Sigma a^2y'z'\,\Sigma\frac{g-h}{yz'-y'z}=\Sigma x'\begin{vmatrix} \dfrac{1}{yz'-y'z} & \dfrac{1}{zx'-z'x} & \dfrac{1}{xy'-x'y} \\ f-\rho & g-\rho & h-\rho \\ bc\cos Ax' & ca\cos By' & ab\cos Cz' \end{vmatrix}.$$

In Ex. 2 we have $\Sigma p=0$ instead of equation (2). Therefore equation (5) becomes $\Sigma px' \cot A=0$, and the equation of the axis is

$$\Sigma \frac{y' \cot B - z' \cot C}{yz' - y'z} = 0 \quad \text{or} \quad \Sigma \frac{v \cot C - w \cot B}{vy - wz} = 0.$$

In Ex. 3, $\Sigma p(yz'-y'z)^2=0$, where $\Sigma p=0$.

Here coefficient of x^2 is $qz'^2+ry'^2$ and of yz is $-2py'z'$.

Condition for perpendicular lines is

$$\Sigma a^2(qz'^2+ry'^2)+2\Sigma py'z'bc\cos A=0. \quad \ldots\ldots\ldots\ldots\ldots\ldots(6)$$

Coefficient of p is $\quad b^2z'^2+c^2y'^2+y'z'(b^2+c^2-a^2)$,

which $\quad =(b^2z')+c^2y')(y'+z')=a^2y'z' \quad$ or $\quad (c^2y'+b^2z')\Sigma x'-\Sigma a^2y'z'$.

Therefore condition (6) becomes, since $\Sigma p=0$, $\Sigma(c^2y'+b^2z')p=0$, and the equation of the axes is in this case

$$\begin{vmatrix} (yz-y'z)^2 & (zx'-z'x)^2 & (xy'-x'y)^2 \\ c^2y'+b^2z' & a^2z'+c^2x' & b^2x'+a^2y' \\ 1 & 1 & 1 \end{vmatrix}=0.$$

In Ex. 1 we have $x'=y'=z'$, and so the axes are $\Sigma(b^2-c^2)(y-z)^2=0$, or this result may be obtained directly by the same method.

Equi-conjugate Diameters.

These form a harmonic pencil with the axes. Also the equation of the lines joining A to the points of intersection of two lines with BC is obtained by putting $x=0$ in the equation of the lines. Thus, in Ex. 1, the pair of lines

$$p(y-z)^2+yz^2+ry^2=0,$$

where $\quad \Sigma p=0 \quad$ and $\quad (b^2-c^2)(y-z)^2+(c^2-a^2)z^2+(a^2-b^2)y^2=0,$

i.e. $\quad qy^2+2pyz+rz^2=0 \quad$ and $\quad (c^2-a^2)y^2+2(b^2-c^2)yz+(a^2-b^2)z^2=0,$

are to form a harmonic pencil;

$$\therefore\ q(a^2-b^2)+r(c^2-a^2)=2p(b^2-c^2)\equiv -2(q+r)(b^2-c^2);$$

$$\therefore\ q(a^2+b^2-2c^2)=r(c^2+a^2-2b^2);$$

$\therefore$ equation of the equi-conjugates is $\Sigma(b^2+c^2-2a^2)(y-z)^2=0$.

In Ex. 3 the result of putting $x=0$ in the general equation of the conjugate diameters is $\quad p(yz'-y'z)^2+qz'^2x'^2+rx'^2y^2=0, \quad \Sigma p=0,$

i.e. $\quad y^2(pz'^2+rx'^2)-2py'z'yz+z^2(py'^2+qx'^2)=0.$

If then p, q, r are the coefficients for the axes and α, β, γ those for the equi-conjugates, the condition for a harmonic pencil is

$$(pz'^2+rx'^2)(\alpha y'^2+\beta x'^2)+(py'^2+qx'^2)(\alpha z'^2+\gamma x'^2)=2\alpha py'^2z'^2$$

or

$$\Sigma\alpha(ry'^2+qz'^2)=0.$$

$\therefore$ the required equation of the equi-conjugates is

$$\begin{vmatrix} (yz'-y'z)^2 & (2x'-z'x)^2 & (xy'-x'y)^2 \\ ry'^2+qz'^2 & pz'^2+rx'^2 & qx'^2+py'^2 \\ 1 & 1 & 1 \end{vmatrix},$$

where $\quad \Sigma p(c^2y'+b^2z')=0 \quad$ and $\quad \Sigma p=0.$

In the general case and with similar notation, we obtain

$$pzx'(-x'y)+q(-x'y)(yz'-y'z)+r(yz'y'z)(z'x')=0$$

as the result of putting $x=0$ in the general equation,

i.e. $\quad qz'y^2-(-px'+qy'+rz')yz+ry'z^2=0.$

Result of making the harmonic pencil is

$$2(q\gamma + r\beta)y'z' = (-px' + qy' + rz')(-\alpha x' + \beta y' + \gamma z'),$$

$$i.e.\quad \Sigma\alpha x'(-px' + qy' + rz') = 0.$$

Hence the equation of the equi-conjugates is

$$\begin{vmatrix} (zx' - z'x)(xy' - x'y) & (xy' - x'y)(yz' - y'z) & (yz' - y'z)(zx' - z'x) \\ x'(-px' + qy' + rz') & y'(px' - qy' + rz') & z'(px' + qy' - rz') \\ f - \rho & g - \rho & h - \rho \end{vmatrix} = 0,$$

where $\Sigma a^2 y'z'\Sigma p = \Sigma bc \cos Apx'\Sigma x'$ and $\Sigma(f - \rho)p = 0.$

In the case of Ex. 2 this becomes

$$\begin{vmatrix} (zx' - z'x)(xy' - x'y) & (xy' - x'y)(yz' - y'z) & (yz' - y'z)(zx' - z'x) \\ x'(-px' + qy' + rz') & y'(px' - qy' + rz') & z'(px' + qy' - rz') \\ 1 & 1 & 1 \end{vmatrix} = 0,$$

where $\Sigma bc \cos Apx' = 0$, *i.e.* $\Sigma a^2(-px' + qy' + rz') = 0$ and $\Sigma p = 0.$

Now $\quad \Sigma x'(y' + z')(-px' + qy' + rz') \equiv 2x'y'z'\Sigma p = 0.$

Hence multiplying the columns in the above determinant first by $a^2y'z'$, etc., and then by $y' + z'$, etc., and adding in each case, we have

$$\begin{vmatrix} \Sigma a^2y'z'/(yz' - y'z) & \Sigma(y' + z')/(yz' - y'z) & 1/(xy' - x'y) \\ 0 & 0 & z'(\quad) \\ a^2y'z' + b^2z'x' + c^2x'y' & 2(x' + y' + z') & 1 \end{vmatrix} = 0,$$

$$i.e.\quad \Sigma a^2y'z' \Sigma \frac{y' + z'}{yz' - y'z} = 2\Sigma x' \Sigma \frac{a^2y'z'}{yz' - y'z}$$

or

$$\Sigma a^2 u \Sigma \frac{v + w}{vy - wz} = 2\Sigma vw \Sigma \frac{a^2}{vy - wz}.$$

Compare with the equation of the equi-conjugates in the general Cartesian conic, viz.:

$$(a + b)(bX^2 - 2hXY + aY^2) = 2(ab - h^2)(X^2 + Y^2).$$

N. M. Gibbins.

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154. From the notes in Monmouth's pocket-book found on his person when he was taken prisoner after Sedgemoor, 1685:

"To know the sum of numbers before they be writ doun.

Astrological rules in French for finding anything required.

A Planetary Wheel, dated 1680, to show life or death in case of illness, also happiness and adversity.

Arithmetical table of the number seven from 1 to 37."

155. Merlin's Steelyard given me.—Entry in Johnson's *Diary of Prayers and Meditations*, Birkbeck Hill, i. 106.

156. *The Quarterly Review*, vol. 68, p. 366, states in a review of Carleton's *Sketches of the Irish Peasantry* that the Irish peasantry are especially fond of a mathematics. "In carrying on the late survey of Ireland, boys were found in abundance to calculate at a halfpenny a mile."

157. I vow and affirm your tailor must needs be an expert geometrician; he has the longitude, latitude, altitude, profundity, every dimension of your body so exquisitely . . . as if your tailor were deep read in astrology, and had taken measure of your body with a Jacob's staff, an ephemerides.—Massinger's *Fatal Dowry*, IV. i.

A CURIOUS PROPERTY OF NUMBERS.

THE difference between any number and another formed by reversing its digits has certain interesting properties, which, I believe, have not been explicitly noticed before, and these properties determine the necessary and sufficient conditions that a number may be expressed as such a difference.

To begin with, let us take a number having an even number of digits and denote it by

$$N = a_1 a_2 a_3 \dots a_r \dots a_p b_p b_{p-1} \dots b_3 b_2 b_1,$$

where a_p, b_p are the middle digits, b_1 the units digit and $a_1 > b_1$. Let us assume further that $a_r \neq b_r$ for any value of r (from 1 to p). Reversing the digits, we form another number

$$N_1 = b_1 b_2 b_3 \dots b_r \dots b_p a_p a_{p-1} \dots a_3 a_2 a_1.$$

Let $\quad R = N - N_1 = c_1 c_2 c_3 \dots c_r \dots c_p d_p d_{p-1} \dots d_r \dots d_3 d_2 d_1.$

A little consideration will show that

$$\begin{aligned}
&\text{(i)} \quad c_r + d_r = 10, \text{ if } a_{r-1} < b_{r-1} \text{ and } a_{r+1} > b_{r+1};\\
&\text{(ii)} \quad c_r + d_r = 8, \text{ if } a_{r-1} > b_{r-1} \text{ and } a_{r+1} < b_{r+1};\\
&\text{(iii)} \quad c_r + d_r = 9, \text{ if } a_{r-1} < b_{r-1} \text{ and } a_{r+1} < b_{r+1};\\
&\qquad\qquad\qquad\quad \text{or if } a_{r-1} > b_{r-1} \text{ and } a_{r+1} > b_{r+1};
\end{aligned}$$

and by the Law of Converses, the converses of the above results are also true.

As we have exhausted all the possible cases, it follows that $c_r + d_r$ cannot take any other value except 8, 9 or 10.

Hence, we may enunciate the following property of R, viz.:

The sum of the two middle digits, as well as that of any pair of digits equidistant from them, is always 8 or 9 or 10. ....(1)

Since $a_1 > b_1$, and therefore $b_1 < a_1$, we infer $d_1 = 10 + b_1 - a_1$ and $c_1 = a_1 - b_1$ or $a_1 - b_1 - 1$, so that $c_1 + d_1$ can never be 8. ....(2)

Similarly, we can show that $c_2 + d_2 \neq 10$. ....(3)

Now, let us put $s_r = c_r + d_r$ and form the series $s_1, s_2, s_3, \dots s_r, \dots s_p$.

If $s_r = 10$ and $a_r > b_r$, then

$$s_{r-1} = \overline{10 + a_{r-1} - b_{r-1}} + \overline{b_{r-1} - a_{r-1}}$$

or

$$\overline{10 + a_{r-1} - b_{r-1}} + \overline{b_{r-1} - a_{r-1} - 1};$$

$$\therefore \; s_{r-1} = 10 \text{ or } 9.$$

Similarly $\qquad s_{r+1} = 9 \text{ or } 8.$

But if $s_r = 10$ and $a_r < b_r$, we get

$$s_{r-1} = 9 \text{ or } 8 \quad \text{and} \quad s_{r+1} = 10 \text{ or } 9.$$

From the above, we gather that, if $s_r = 10$, its predecessor and successor cannot both be either 8 or 10. ....(4)

In the same way, if $s_r = 8$, its predecessor and successor cannot both be either 8 or 10. ....(5)

Again, if we form the series of alternate terms

$$s_1, s_3, s_5, \dots \quad \text{and} \quad s_2, s_4, s_6, \dots,$$

we observe that the terms, if any, in the first series, barring 9's, should be of the type

$$10, 8, 10, 8, \dots,$$

while those of the second series are, in order,

$$8, 10, 8, 10, \dots.$$

Thus, if we omit the nines in the series of alternate terms, the first series begins with 10 and the second begins with 8, and the terms are alternately 10 and 8. ....(6)

If we denote the number of 8's in the s-series by 'x' and the number of 10's by 'y,' it is easily seen from the preceding property that $x \sim y = 0$ or 1 or 2. But, since R is divisible by 9, $x \sim y = 0 \pmod 9$.

Hence $x = y$, *i.e.* the number of 8's in the s-series, is equal to the number of 10's. ....(7)

It is also obvious that the sum of the digits in R is equal to

$$8x + 10y + (p - x - y)9,$$

i.e. $9p$. ....(8)

Lastly, one notes that if there should be no 9's at all in the series

$$s_1, s_2, \ldots s_r \ldots s_p,$$

it should be identical with the series 10, 8, 8, 10, 10, 8, 8, 10, 10, ... But all the terms in the s-series cannot be nines, nor eights, nor tens. ....(9)

Of the nine properties, noted above, of the number R, (1), (4), (5), and (6) give the necessary and sufficient conditions that R can be expressed as the difference between a number and its invert; and if these conditions be satisfied the number of ways in which R can be so expressed can be easily calculated. Thus, when $R = 1761638418$, we can find N as follows.

Forming the s-series, we get 9, 8, 10, 9, 9, which obviously satisfies all the conditions laid down above. Hence, a solution is possible.

Since $s_1 = 9$, $a_2 < b_2$; and $s_2 = 8$ shows that $a_3 < b_3$, while from $s_3 = 10$, we gather $a_4 > b_4$ and $s_4 = 9$ leads us to the fact $a_5 < b_5$.

Now, we can easily form the following indeterminate equations

$$a_1 - b_1 = 2,$$
$$b_2 - a_2 = 2.$$
$$b_3 - a_3 = 4,$$
$$a_4 - b_4 = 2,$$

and $$b_5 - a_5 = 4.$$

Recollecting that all the digits can take all integral values from 0 to 9, except a, which cannot take the value 0, we see that the total number of solutions for N is $(10 - 2)(10 - 2)(10 - 4)(10 - 2)(10 - 4)$,

i.e. 18,432.

One point more. Given the number of terms in the s-series to be n, the total number of permutations of its terms can be shown to be 2^{n-1}, due regard being paid to the restricting conditions.

It will be interesting to calculate in how many ways the terms of the s-series can be arranged, if we previously know how many 8's, 9's and 10's there are in it.

In fine, it may be observed that there is no loss of generality in taking N with an even number of digits and also assuming that the equidistant digits are unequal. For, when N contains an odd number of digits, the middle digits in N and N_1 will be equal, and whenever equal digits occur such as $a_r = b_r$, the corresponding digit of the remainder R will be 0 or 9, and the rest of the digits in R will be unaltered even if the equal digits be removed. Thus, in such cases, we may omit the pairs of equal equidistant digits in N and N_1, and proceeding with the subtraction, we get the same digits in the remainder as before with the omission of the 0's and 9's, which correspond to the equal digits.

We can also easily determine whether N should contain pairs of equal equidistant digits, by inspection of the digits in R, if we note the following rule, which can be easily proved: If a pair of equidistant digits (say c_r, d_r) in R be either two 0's or two 9's, then they correspond to equal digits in N; but if it be 0 and 9, they correspond to equal digits only when the predecessor, if any (barring nines), of s_r ($= 9$) in the series of alternate terms to which s_r belongs is 10 or 8, according as c_r is 0 or 9.

For example, let $R = 19897030098$. Here the middle digit is 0 as it ought to be, and the two equidistant 9's show that there are equal digits in N; and we have for the s-series 9, 8, 9, 10, omitting the term 18. The s-series can be broken up into two series 9, 9 and 8, 10 to, the former of which 9 ($=s_4$) belongs, and this has no predecessor barring nines. So 9 ($=c_4$) and 0 ($=d_4$) correspond to equal equidistant digits in N. Omitting therefore these digits, which betray the equal digits in N, we may write $R' = 1.8.7 = 3.0.8$, indicating the lacunae by dots. Proceeding as before, we can determine the N' corresponding to R'.

Thus $N' = 3.8.4.7.9.1$, and we may get N from this by putting in any pairs of equal equidistant digits in the lacunae indicated by dots. Hence

$$N = 35874877951.$$

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158. The Poetry of Mathematics (Edgar Quinet, *Œuvres complètes*, Hachette). It is a great mistake to suppose that one may not be enthusiastic about mathematical truths. The contrary is the fact. A problem in analysis, a great effort in calculation, or the solution of an equation demand from a Kepler, a Galileo, a Newton or an Euler, as much intuition or spontaneous inspiration as one of Pindar's *Olympiads*.

Those pure and incorruptible formulae, which were before the world, and which will be after it, which dominate all time and space, which are, so to speak, an integral part of God Himself, which will survive the ruin of all the universes, bring the mathematician who is worthy of the name into intimate communion with the divine thought. In these immutable truths he tastes the purest creative work, he prays in his own tongue like one of old: "Silence! and we shall hear the murmuring of the Gods." I loved like a Pythagorean the incorruptible purity of geometry. The luminous and mysterious language of algebra fascinated me. What I marvelled at most in this language is that, disdaining the individual truth, it consented to express and to articulate none but such as are general and universal. To it I attributed a pride which I refused to human speech, and from this point of view algebra seemed to me part of a celestial tongue, the language of the god of mind. I also appreciated the style appropriate to algebra: I was struck with the art with which the mathematicians lay aside, reject, eliminate all that is unnecessary and explain the absolute in the smallest number of terms, and yet preserve in the arrangement of those terms a choice, a parallelism, and a symmetry which seemed to be the elegance inherent in a beauty of thought that is eternal. And this suggested to my mind the idea of style as brief, condensed, and radiant, and I began to correct my long, dragging, sprawling sentences.

But if Algebra struck me, I was dazzled by Geometry. Conic sections threw me into a state of wonderment even more than did "The Thousand and One Nights." The surprising properties of the curves astounded me. From marvel to marvel they led me till I thought I must be on the threshold of the laboratory of creation.

And as I began to understand, I felt myself more and more a spark in the flame of God Himself. The idea, the possibility of expressing a line or a curve by algebraical terms, or by an equation, seemed to me as grand as the *Iliad*. When I saw the equation functioning, and solving itself, as it were, in my very hands, and bursting into an infinite number of equally indubitable, eternal, and resplendent truths, I seemed to have the talisman which would open to me the gates of all the mysteries.

MATHEMATICAL NOTES.

649. [D. 2.] *Productive Fractions.*

1. A series of *productive* fractions is one of which every term is the product of the preceding one and another fraction, called the distinctive fraction.

Thus $\frac{1}{2}+\frac{1}{2}\cdot\frac{1}{3}+\frac{1}{2}\cdot\frac{1}{3}\cdot\frac{1}{4}+\frac{1}{2}\cdot\frac{1}{3}\cdot\frac{1}{4}\cdot\frac{1}{5}$ is a productive fraction which can be written $\frac{1}{2}/\frac{1}{3}/\frac{1}{4}/\frac{1}{5}\ldots$.

When a term is negative, the negative sign is added above its distinctive fraction, thus $\frac{1}{2}+\frac{1}{2}\cdot\frac{1}{3}-\frac{1}{2}\cdot\frac{1}{3}\cdot\frac{1}{4}\ldots$ is written $\frac{1}{2}/\frac{1}{3}/\frac{\bar{1}}{2}\ldots$.

When the distinctive fractions recur, a vinculum is placed over those which do so, and the letter R is written after them, thus

$$\tfrac{1}{8}+\tfrac{1}{8}\cdot\tfrac{1}{8}+\tfrac{1}{8}\cdot\tfrac{1}{8}\cdot\tfrac{1}{8}\ldots \text{ is written } \overline{\tfrac{1}{8}}/R.$$

And $\frac{1}{80}+\frac{1}{80}\cdot\frac{1}{11}+\frac{1}{80}\cdot\frac{1}{11}\cdot\frac{1}{20}+\frac{1}{80}\cdot\frac{1}{11}\cdot\frac{1}{20}\cdot\frac{1}{11}+\frac{1}{80}\cdot\frac{1}{11}\cdot\frac{1}{20}\cdot\frac{1}{11}\cdot\frac{1}{20}\ldots$

is written $$\tfrac{1}{80}/\overline{\tfrac{1}{11}/\tfrac{1}{20}}/R.$$

2. Suppose p/q is any fraction, and that $r, s, \ldots$ are any numbers greater than q, then it can easily be shown that

$$p/q=p/r\Big/\frac{r-q}{q}=p/r\Big/\frac{r-q}{s}\Big/\frac{s-q}{q}=p/r\Big/\frac{r-q}{s}\Big/\frac{s-q}{t}\Big/\frac{t-q}{q}.$$

If $r, s, t, \ldots$ are chosen so as to be multiples of the corresponding numerators, many useful series can be obtained, thus:

$$\tfrac{1}{73}=\tfrac{1}{80}/\overline{\tfrac{1}{11}/\tfrac{1}{20}}/R.$$

By means of this series, $\frac{1}{73}$ can be quickly evaluated.

$$\begin{array}{r}
1\cdot 000\ 000\ 000\ 0\\
\cdot 012\ 500\ 000\ 0\\
\cdot 001\ 136\ 363\ 6\\
\cdot 000\ 056\ 818\ 2\\
5\ 165\ 3\\
258\ 3\\
23\ 5\\
1\ 2\\
1\\
\hline
0\cdot 013\ 698\ 630\ 2
\end{array}$$

When a fraction is a multiplier, the product in decimals can be obtained easily by the use of productive fractions.

Multiply £134. 16*s*. 7*d*. by $\frac{37}{126}$.

$$\tfrac{37}{126}=\tfrac{37}{148}/\tfrac{22}{132}/\tfrac{6}{126}=\tfrac{1}{4}/\tfrac{1}{6}/\tfrac{1}{21}.$$

$$\begin{array}{ll}
4 & \underline{\pounds 134\cdot 829}\\
6 & \quad 33\cdot 707\\
21 & \quad\ \ 5\cdot 618 \div 3,\ 1\cdot 873\\
 & \quad\ \ \underline{\ \ \cdot 268}\\
 & \pounds 39\cdot 593 = \pounds 39.\ 11s.\ 10\tfrac{1}{4}d.
\end{array}$$

3. Productive fractions can be applied to surds.

p is any number, and $\sqrt{p}$ lies between a and $(a+1)$, $p-a^2=d$.

If $2l_1, 2l_2, 2l_3, \ldots$ are the denominators of the distinctive fractions $(l_1=a)$ then

$$\sqrt{p}=a+\frac{d}{2a}\Big/\frac{\bar{d}}{2(d+2a^2)}\Big/\frac{\bar{d^2}}{2(d^2+2^3a^2p)}\Big/\frac{\bar{d^4}}{2(d^4+2^5a^2l_2^2p)}\Big/\frac{\bar{d^8}}{2(d^8+2^7a^2l_2^2l_3^2p)},$$

the series continuing to infinity.

After a few terms the series converges with amazing rapidity. Thus

$$\sqrt{5}=2+\frac{1}{4}\Big/\frac{\bar{1}}{2.9}\Big/\frac{\bar{1}}{2.161}\Big/\frac{\bar{1}}{2.51841}\Big/\frac{\bar{1}}{2.5374978561}\Big/\dots.$$

Calling the fractional terms T_1, T_2, ..., we have

$$T_1=0{\cdot}25,$$
$$T_2=0{\cdot}013{,}8\ 88{,}8\ 88{,}8\ 88{,}8,$$
$$T_3=\tfrac{1}{322}T_2=0{\cdot}000\ 043\ 133\ 195\ 31,$$
$$T_4=\tfrac{1}{103682}T_3=0{\cdot}000\ 000\ 000\ 416\ 01.$$
$$T_2+T_3+T_4=0{\cdot}013\ 932\ 022\ 500\ 2,$$
$$T_1=0{\cdot}25;$$
$$\therefore\ \sqrt{5}=2{\cdot}236067977499\dots.$$

R. W. M. Gibbs.

650. [V. 1. μ.] *Mathematics on the Gramophone.*

Suppose that the record revolves n times a second, that the width of the groove in which the needle runs is d feet, and that the needle when the tune starts is a feet from the centre; then t seconds later its distance r, and its velocity along the groove v are approximately given by

$$r=a-ndt,$$
$$v=2\pi rn=2\pi an-2\pi dn^2t,$$

so there is uniform retardation or $f=-2\pi dn^2$. Now, if s is the length of groove traversed, sd is the area the needle has covered on the record and is $\pi a^2-\pi r^2$, so

$$S=\frac{\pi a^2-\pi r^2}{d}=\frac{4\pi^2n^2a^2-4\pi^2n^2r^2}{4\pi n^2d}=\frac{u^2-v^2}{-2f},$$

i.e. the formula for uniform acceleration

$$v^2-u^2=2fs$$

is established.

From this point of view it is a pity that manufacturers do not make the needle start in the middle and work outwards. In such a case, if ω is the angular velocity of the record, θ the angle turned through in time t,

$$r=\frac{\theta}{2\pi}d,\quad v=r\omega,\quad s=\frac{\pi r^2}{d}=\frac{d}{4\pi}\theta^2,$$

and if ω is constant, $=k$,

$$\theta=kt,\quad v=\frac{dk^2}{2\pi}t,\quad s=\frac{dk^2}{4\pi}t^2=\tfrac{1}{2}ft^2.$$

We may, however, play in a futurist manner and allow ω to vary. If, for instance, $\theta=k\sqrt{t}$, then $s=\frac{dk^2}{4\pi}t$, so v is a constant, $\frac{dk^2}{4\pi}$, and $\omega=\frac{v}{r}=\frac{k}{2\sqrt{t}}$, thus establishing the fact that $\int\frac{k}{2\sqrt{t}}dt=k\sqrt{t}$, which throws light on the motion of a particle attracted under the law of the inverse cube.

Generally the gramophone shows $\int f(t)f'(t)dt=\frac{[f(t)]^2}{2}$.

The change caused by taking the groove—more accurately—to be the spiral $r=a\theta$, where $a=\frac{d}{2\pi}$, is striking. For then, if the record turns with uniform angular velocity ω, $v=a\omega\sqrt{\theta^2+1}$, *f along the groove* $=\frac{a\omega^2\theta}{\sqrt{\theta^2+1}}$, and $s=\frac{a}{2}(\theta\sqrt{\theta^2+1}+\sinh^{-1}\theta)$.

The area covered by the needle is, however, less bizarre, being $\pi(r^2 - rd)$.

In view of this it is perhaps surprising that the first approximation gives such an excellent result. C. H. HARDINGHAM.

651. [L^2. 2. d.] *To find the angle between the lines in which a cone is cut by a plane through its vertex.*

The following method is, I believe, new.

Let the cone be

$$S \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0,$$

and the plane $l_0x + m_0y + n_0z = 0$, where $l_0^2 + m_0^2 + n_0^2 = 1$.

Let S_{11}, S_{22}, S_{00} be the results of substituting (l, m, n), etc., in S, and let

$$S_{12} \equiv \Sigma a l_1 l_2 + \Sigma f(m_1n_2 + m_2n_1) \equiv S_{21}.$$

	ξ	η	ζ
x	l_1	l_2	l_0
y	m_1	m_2	m_0
z	n_1	n_2	n_0

Changing the axes according to the scheme, the equations of cone and plane become

$S_{11}\xi^2 + S_{22}\eta^2 + S_{00}\zeta^2 + 2S_{20}\eta\zeta + 2S_{01}\zeta\xi + 2S_{12}\xi\eta = 0$ and $\zeta = 0$,

and the angle required is now evidently given by

$$\theta = \tan^{-1}(2\sqrt{S_{12}^2 - S_{11}S_{22}})/(S_{11} + S_{22}). \quad \text{.........................(1)}$$

It is obvious that $S_{11} + S_{22} = (a + b + c) - S_{00}$; it remains therefore to express $S_{12}^2 - S_{11}S_{22}$ in terms of the original coefficients.

Let $\Sigma_0 \equiv \begin{vmatrix} a & h & g & l_0 \\ h & b & f & m_0 \\ g & f & c & n_0 \\ l_0 & m_0 & n_0 & 0 \end{vmatrix}$ and $\sigma \equiv \begin{vmatrix} l_1 & l_2 & l_0 \\ m_1 & m_2 & m_0 \\ n_1 & n_2 & n_0 \end{vmatrix} = \pm 1.$

Then $$\Sigma_0\sigma = \begin{vmatrix} al_1 + hm_1 + gn_1 & hl_1 + bm_1 + fn_1 & gl_1 + fm_1 + cn_1 & 0 \\ al_2 + hm_2 + gn_2 & hl_2 + bm_2 + fn_2 & gl_2 + fm_2 + cn_2 & 0 \\ al_0 + hm_0 + gn_0 & hl_0 + bm_0 + fn_0 & gl_0 + fm_0 + cn_0 & 0 \\ l_0 & m_0 & n_0 & 1 \end{vmatrix};$$

and multiplying again, $$\Sigma_0\sigma^2 = \begin{vmatrix} S_{11} & S_{12} & S_{10} \\ S_{12} & S_{22} & S_{20} \\ 0 & 0 & 1 \end{vmatrix},$$

or $$\Sigma_0 = S_{11}S_{22} - S_{12}^2.$$

Hence $$\theta = \tan^{-1}(2\sqrt{-\Sigma_0})/(\Sigma a - S_{00}).$$

R. C. G. HOWLAND.

652. [V. a. ι.] *An Example of Abridged Notation.*

A quadrilateral $ABCD$, in which $AB + CD = AC + BD$, possesses an inscribed circle; if, on the other hand, $AB + BC = CD + DA$ there exists an escribed circle.

The former condition is satisfied by the quadrilateral formed by the foci of a hyperbola and any two points on the curve; the latter by the foci and any two points of an ellipse. And in each case the pole of the chord joining the two points is the centre of the circle.

The following analytical proof is interesting as resting entirely on the projective definition of foci and leading to a well-known property.

Let P, Q be the foci, B, C two points on the conic, A the pole of BC, and let $A = 0$, $B = 0$ be the tangential equations of the respective points. Then the equation of the conic may be written in either of the forms

$$PQ + \lambda(l^2 + m^2) = 0, \quad BC + A^2 = 0.$$

Hence, absorbing a constant in the equations of the points, we have the identity

$$PQ + \lambda(l^2 + m^2) \equiv BC + A^2.$$

Transposing, we obtain

$$A^2 - \lambda(l^2 + m^2) \equiv PQ - BC.$$

The left-hand side equated to 0 gives a circle with centre A; the right-hand side equated to 0 gives a conic touching PB, PC, QB, QC; which proves the proposition.

Note that this gives an independent proof that the tangent bisects the exterior angle PBQ.

Various special cases arise; as when PC passes through Q, in which case the circle has Q for its point of contact. Again, if Q is at an infinite distance, we have the proposition that, if A be the pole of a chord BC of a parabola whose focus is P, then A is the centre of a circle touching PB, PC and the diameters through B and C; this contains several theorems in the geometry of the parabola.

We may now generalise by replacing PQ by an undegenerate conic Σ'. The enunciation now is: If from points BC on a conic Σ four tangents be drawn to a confocal Σ', they are touched by a circle whose centre is the pole of BC w.r.t. Σ.

This proves independently that the tangents from any point B to a system of confocals form an involution, the double rays being the tangent and normal at B to the confocal through B.

Generalising further, we replace BC by a conic σ; the identity now becomes

$$(\Sigma \equiv)\Sigma' + \lambda(l^2 + m^2) \equiv \sigma + A^2,$$

and transforms into $\qquad A^2 - \lambda(l^2 + m^2) \equiv \Sigma' - \sigma;$

and we have the proposition: If a conic σ have double contact with any confocal of a conic Σ', the four common tangents of Σ' and σ touch a circle whose centre is the pole of the chord of contact.

Extensions to three dimensions can also be made. R. C. G. HOWLAND.

653. [K[1]. 8. b.] *Ptolemy's Theorem.*

The following two proofs of Ptolemy's theorem are new and interesting. Of them, one does not involve the principle of ratios and similar triangles, while the other shows the connection of Ptolemy's theorem with Stewart's Theorem.

First Proof: Let $ABCD$ be the cyclic quadrilateral. Construct rectangles $BHEC$ and $DCFG$ on the sides BC, CD respectively outside the quadrilateral, making $CE = AD$ and $CF = AB$. Produce GD and HB to meet at K. Then, obviously, K lies on the circum-circle of the quadrilateral, and

$$B\hat{K}D = E\hat{C}F = B\hat{A}D.$$

Also, KC is a diameter and KA is perpendicular to AC.

The triangles ABD and CFE are easily seen to be congruent, so that

$$A\hat{D}B = C\hat{E}F \quad \text{and} \quad BD = EF.$$

But $A\hat{D}B = A\hat{C}B$ in the same segment.

$\therefore$ $A\hat{C}B + C\hat{E}F = 90°$, and hence AC is perpendicular to EF.

Since KA and EF are both perpendicular to AC, KA is parallel to EF.

$$\therefore \triangle KEF = \triangle AEF.$$

Taking away from both $\triangle ECF$, we have

$$\triangle KCE + \triangle KCF = \triangle ACE + \triangle ACF,$$

i.e. $\qquad \tfrac{1}{2}BC \cdot EC + \tfrac{1}{2}CF \cdot CD = \tfrac{1}{2}AC \cdot EF,$

i.e. $\qquad BC \cdot AD + AB \cdot CD = AC \cdot BD.$ $\qquad$ Q.E.D.

Second Proof: Let the diagonals AC, BD cut at X.
Applying Stewart's Theorem to the triangle ABD, we get

$$XD \,.\, AB^2 + BX \,.\, AD^2 = BD(AX^2 + BX \,.\, XD). \qquad \text{(I)}$$

From the properties of the cyclic quadrilateral, we have

(i) $BX \,.\, XD = AX \,.\, XC$, so that $AX^2 + BX \,.\, XD = AX \,.\, AC$,

and (ii) the triangles DXC and AXB are similar, so that

$$\frac{XD}{XA} = \frac{CD}{AB},$$

and in the same way, from the similar triangles BXC and AXD,

$$\frac{BX}{AX} = \frac{BC}{AD}.$$

Dividing (I) by AX, we have

$$\frac{XD}{AX} \,.\, AB^2 + \frac{BX}{AX} \,.\, AD^2 = \frac{BD}{AX}(AX^2 + BX \,.\, XD),$$

i.e.
$$\frac{CD}{AB} \,.\, AB^2 + \frac{BC}{AD} \,.\, AD^2 = \frac{BD}{AX} \,.\, AX \,.\, AC,$$

i.e.
$$AB \,.\, CD + BC \,.\, AD = BD \,.\, AC. \qquad \text{Q.E.D.}$$

Note.—By reversing the above steps, we can deduce Stewart's theorem from Ptolemy's theorem.

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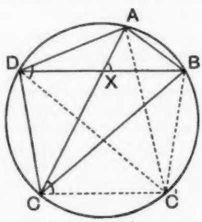
654. [K^1. 8. b.] *Proof of Ptolemy's Theorem.*

Draw chord CC' parl. to BD.
Join BC', $C'D$.
Then quadl. $ABC'D$ = quadl. $ABCD$
= $\triangle$ with sides equal to AC, BD inclined at the same angle X.

$$BC' = CD; \quad C'D = BC.$$

Also
$$A\hat{D}C' = A\hat{C}C' = X,$$

and therefore $A\hat{B}C$ is supplementary to X.



But triangles with equal or supplementary angles are in the ratio of the rectangles contained by the sides about them;

$$\therefore \; AD \,.\, DC' : AC \,.\, BD = \triangle ADC' : \text{quadl. } ABC'D$$

and
$$AB \,.\, BC' : AC \,.\, BD = \triangle ABC' : \text{quadl. } ABC'D;$$

$$\therefore \; AD \,.\, DC' + AB \,.\, BC' = AC \,.\, BD,$$

i.e.
$$AB \,.\, BC + AB \,.\, CD = AC \,.\, BD.$$

E. M. LANGLEY.

REVIEWS.

First Ideas of Geometry. By G. St. L. Carson and D. E. Smith. Pp. 90. 1s. 6d. 1922. (Ginn & Co.)

From 1871 for about thirty years the A.I.G.T. strove with little apparent success for such modifications of examination regulations as would permit the use of text-books on Geometry other than the "Elements" of Euclid. Then they were aided by powerful allies in Prof. Perry and others interested rather in the applications of Geometry to Engineering than in the want of the teacher, and momentous changes followed. The use of text-books other than the famous "Elements" was not only granted to teachers, but almost forced upon them by the new regulations issued by universities and other examining bodies. To supply the void created by the dethronement of "Euclid," it was natural that many an aspirant should produce some sort of substitute; it was natural also that, in the words of the authors of the small introduction before us, "many of those who advocated this departure rushed to the confusion which so often follows such a change . . . the trouble lay not with the spirit of Euclid but with the failure of most teachers"—we would add "of most examining bodies"—"to adapt his spirit to modern schools and conditions." But something more was wanted than mere relaxations from pedantic adherence to an ancient text. It had become generally recognised that the pupil before being confronted with severe exercises in Logic should have his mind familiarised with some of the principal ideas of position, shape and size about which he will have to reason. It is with this aim that the authors prefaced their rigorous treatises with these "First Ideas." Notions of symmetry, similarity, slope, etc., are well illustrated through familiar objects. The illustrations are not only good in themselves, but are suggestive to teachers for future development. In the case of similarity, for instance, we get the idea of giving a figure a uniform radial stretch from a point illustrated by the use (i) of an elastic string, (ii) of a simple pantagraph such as could be constructed with strips of cardboard. Elementary Mensuration and Geometrical Drawing receive due attention. A copious collection of exercises for class use sufficient both for a first course and revision is supplied. The work can be recommended as a sound introduction to the formal treatise by the authors or to any other text-book covering the same ground—the old Bks. I.-IV., VI.

Edward M. Langley.

Bell's Mathematical Tables, together with a collection of Mathematical Formulæ, Definitions and Theorems. By L. Silberstein. Pp. xii + 250. 16s. 1922. (Bell & Sons.)

Publishers must be assumed to know their own business, but we cannot understand why this valuable book must run the risk of being dismissed as an expensive substitute for Chambers when it might have compelled attention as a "Synopsis of Applicable Mathematics, with Tables".

Part I. of the book is occupied by ordinary tables of logarithms, of logarithmic sines and tangents, and of reciprocals, together with a table to assist the passage between common logarithms and natural logarithms, and one for converting degrees into radians. The type used for these tables is not the largest that the size of page would allow, but the smallest that can be read comfortably, so that there is a clear space round every entry, and the danger of taking out the wrong item is reduced to a minimum.

It is on Part II. that the title we have suggested throws the emphasis, but this part is three-quarters of the whole.

The survey of algebra and analysis includes theorems on prime numbers, interpolation, matrices and determinants, algebraic equations, complex numbers, infinite series products and continued fractions, approximation to roots, differentiation and integration (with a table of standard forms), the gamma function, approximate quadrature, Cauchy's theory of residues, spherical zonal and cylindrical harmonics, elliptic integrals, Jacobian and Weierstrassian elliptic functions, and the calculus of probabilities. Inter-

spersed are numerical tables of hyperbolic cosines and sines, of Fresnel's integrals and the error integral, of the gamma function and its logarithmic derivative, of the first seven zonal harmonics and their zeroes, of $J_0(x)$, $J_1(x)$ and $K_0(x)$, of elliptic integrals, and of the common logarithm of $\exp(-\pi K'/K)$ in terms of arc sin k.

In geometry there are formulæ relating to mensuration, to plane and spherical trigonometry, to plane and solid analytical geometry, to the differential geometry of curves and surfaces (the expression for Gaussian curvature in terms of oblique curvilinear coordinates being given), to non-euclidean geometry, and to projective geometry.

Lastly, 23 pages are devoted to vector algebra and analysis, quaternions, and the tensor calculus, and here Dr. Silberstein has gone farther into detail than in other sections, with a view to making his abstracts intelligible to readers to whom the subjects are new. With vectors and tensors he is successful, but not, we think, with quaternions, for in looking for a short cut he has yielded, like Joly, to the temptation to *define* a quaternion as the sum of a scalar and a vector. How can a writer who recognises on one page that "the sum of two vectors" is a meaningless phrase until a definition has been framed assume a few pages later that a vector can be added to a scalar without any definition of addition at all? The merit in Hamilton's work most often overlooked is that he did not dissect a quaternion until, by utilising its properties as an operator (*Lectures*, § 52, summarised on p. xxxvi) or as a quotient (*Elements*, § 202; this is the method adopted by Tait and others), he could do so without violence to logic.

No two writers would agree on every word in a wide survey of mathematics, and we have marked a number of details for criticism; for example, it is arguable that Taylor's theorem, with a remainder, ought to be enunciated definitely as asserting the existence of a fraction θ.

There is the sprinkling of harmless misprints * to which a book of this kind is specially liable—Lionville, on p. 93, is an enemy so familiar as to be an old friend—and the arrangement of type is sometimes clumsy. But the only formula we have noticed that is likely seriously to distress the reader who comes on it for the first time is that on p. 116 for the asymptotic expansion of the error function; in this series, described incorrectly as semi-convergent, summation is for values of κ from 0 to n, and the restriction appears in the form $\kappa<x^2$, which is different psychologically, though not mathematically, from $n<x^2$.

The proof of the pudding is in the eating, but with a good recipe in the hands of a cook of Dr. Silberstein's experience the result cannot be in doubt. We share his regret that there was no room for sections on differential and integral equations, and we ourselves would gladly have sacrificed Part I. to admit them. Dr. Silberstein invites readers to co-operate by sending him suggestions and corrections; if he will reciprocate by issuing his additions always as supplements to previous editions as well as incorporating them in future editions, the book will rapidly be moulded exactly to practical needs. We welcome his assurance that he regards the book as a beginning, and we say gratefully that it is a far better beginning than anyone with experience of the difficulty of leaving things out would have thought possible. E. H. N.

The Principle of Relativity, with Applications to Physical Science. By A. N. Whitehead. Pp. xii + 190. 10s. 6d. net. 1922. (Cambridge University Press.)

La Relativité Restreinte. By G. Fontené. Pp. viii + 158. 1922. (Vuibert.)

Vector Analysis and the Theory of Relativity. By F. D. Murnaghan. Pp. x + 125. 1922. (Johns Hopkins Press.)

At the present time there is a disposition to re-examine critically the arguments by which Einstein is supposed to have shattered the old ideas of space and time. Painlevé has shown that Newtonian ideas about rotation correspond closely to the experimental facts, while the new doctrines, as usually

* Since writing the above, we hear that a list of errata is now included in each copy as issued.

stated, do not. It is true that these doctrines can be restated so as not to conflict with the facts about rational phenomena ; but even then they merely express in a complicated way what the old ideas expressed more simply. In spite of these difficulties and of many others, the achievements of Einstein's theory have led most of us to believe that it contains, at the lowest, a large proportion of truth. It has succeeded in explaining the anomalies in the motion of Mercury and in predicting the deflection of light by the gravitational field of the sun, whereas most of its critics have the serious defect of being merely negative. However, Professor Whitehead is a striking exception. In his *Principle of Relativity* he attempts to combine the positive results of Einstein's theories with something like the older views of space and time. He sees no necessity, at any rate at present, to discard Euclidean geometry in favour of that of Riemann. For him geometry is still geometry and mechanics still mechanics, and he considers the identification of these two sciences as leading to manifest absurdities when applied to such a problem as that of the motion of the moon affected by the sun and the earth. Yet he enunciates definite postulates which can be treated mathematically, and which lead to the usual results concerning the perihelion of Mercury and the deflection of light by a gravitational field. The spectral shift calculated by these methods is seven-sixths of that given by Einstein's methods. Professor Whitehead also deduces two other results concerning spectral lines, the limb effect due to the sun and the doubling or tripling of the lines due to some nebulae. These have been observed, but they have never been explained before. Some calculations on the motion of the moon are still in progress. Finally, he predicts two novel effects due to the interaction of gravitational and electromagnetic fields. These effects are extremely small, but it is to be hoped that experimenters will test these predictions as soon as possible. Even if delicate observations disprove the formulae of this new theory, two other possible theories are briefly indicated which can be developed if necessary.

It would require too much space to give a detailed account of how these calculations are made. Much of the work is very similar to that of Einstein. The most important difference seems to be that Einstein's law of motion is given by the assumption of a stationary *space-time interval*, a semi-geometrical conception, while Professor Whitehead assumes that what he calls the *impetus* is stationary. The definition of impetus involves mass, as well as space and time, and in the general case it contains a term involving electric charge and potential.

In addition to these highly important hypotheses and predictions, which make up Part II. of the book, an exposition of an appropriate theory of tensors is given in Part III. This makes no use of the geometrical ideas of the "fundamental tensor." Part I. is mostly of a philosophical nature, and is rather difficult for mathematicians who have not Professor Whitehead's breadth of knowledge. It is clear that the book deserves very careful examination by all mathematicians, experimental physicists, and philosophers who are interested in relativity.

M. Fontené treats the restricted theory of relativity in what may be called the pre-Einstein fashion, considering one system to be at absolute rest in the ether while another system is in motion. Measurements taken in the second system are termed pseudo-measures. Many new definitions are introduced, but little use is made of them. It is difficult to see the advantages of a new work written from what appears to be an obsolete point of view. The best part of the book is a short appendix giving an outline of the generalised theory.

Professor Murnaghan gives an account of tensor calculus, with a final chapter showing its application to the generalised theory of relativity. The book is by no means a mere transcript of Einstein's papers. It deals more fully than usual with the purely mathematical side of the subject, in spite of the misleading statement that it is addressed to the physicist. The chapter on integral invariants is a novel feature. The remarks (p. 100) on the different meanings of the word *curvature* are valuable, and they might well have been expanded, as the apparent simplicity of this word leads many astray. It is unfortunate that the book should be disfigured by a misprint in the title stamped in gold letters on its back.

H. T. H. Piaggio.

Problems Made Easy for Preparatory Schools and the Lower Forms in Public Schools. With Answers. By R. Tootell. Pp. v+64+11. 3s. net, post free; 2s., without Answers. 1921. (Warren, High Street, Winchester.)

This is a collection of problems of the easiest type and of considerable variety, to be solved by simple or simultaneous equations. The following is an example of the compiler's expository power :

"If two beetles start towards each other, one from the bows and the other from the stern of a ship, at speeds of 2 and 3 yards a minute respectively, they are approaching at the rate of 5 yards a minute. What would be the effect if the ship began to move forward two yards a minute? The beetle at the bows would not be really moving at all, while the one at the stern would be moving forwards at 5 yards a minute towards the first beetle. Hence the rate of approach would be the same as before."

There are some good points about this little compilation, but as some of the usual text-books cover the same ground just as effectively, it is not easy to see the *raison d'être* of this little manual.

Textile Mathematics. Part II. By T. Woodhouse and A. Brand. Pp. 111+Six Tables. 2s. 6d. 1921. (Messrs. Blackie.)

This part treats first of ratio, proportion and variation, with averages and percentages. Many illustrative examples are given, some solved in more than one way, with sample checkings. Then come calculations of "loss and regain," connected with the "conditioning" of textile materials, and the proportions and costs that arise in mixtures of different qualities of materials. About twenty-five pages are then devoted to indices, logarithms and trigonometrical ratios, with the applications of the latter to rope-pulley grooves, etc. Finally, there is a chapter on yarn counts. The book closes with sets of examples on the chapters, lists of useful technical and other data, and six tables of logarithms, anti-logarithms, natural and logarithmic sines and tangents. A special word of praise is deserved for the clear type and the size of the numerals.

Précis d'Arithmétique. By J. Poirée. Pp. v+64. 7 fr. 50. 1921. (Gauthier-Villars.)

The author begins by showing how from the consideration of a collection of objects emerges the notion of the integer, and how the idea of fraction arises from the comparison of magnitudes of the same species.

From the calculation of a fraction to a given number of places comes the definition of the limit of a series of numbers. From square root we are led to the idea of irrationals : progressions are discussed because they are needed later in connection with logarithms. Sufficient examples are given to test the student's grasp of the subject matter. The last sixteen pages form an introduction to the study of the elementary theory of Numbers, which forms an integral part of the arithmetical course in French and Italian schools. Quite three-fourths of the new (6th) edition of Prof. Fazzari's *Elementi di Aritmetica con note storiche* for secondary schools are devoted to easy number theory.

Compound Interest. By A. Skene Smith. Pp. 63. 1s. 6d. net. 1922. (Effingham Wilson.)

Comprehensive Tables of Compound Interest on £1, £5, £25, £50, £75, and £100. By John Wilhelm. Pp. 112. n.p. 1922. (Effingham Wilson.)

This second and enlarged edition deals with the problems that occur in Deferred and Immediate Annuities, Present Value and Amounts, Insurance Premiums, Repayment of Loans, Capitalisation of Rentals and Incomes, etc. Within a small space the author expounds what he calls the "principals" of the subject, and there is little doubt that those who take the trouble to master these pages will not find it difficult "to clearly understand" the ordinary difficulties that arise. Mr. Wilhelm's *Tables* show the accumulations year by year for fifty years at rates of interest from one (progressing by $\frac{1}{4}$) to five per cent. It has also tables of Interest, Rebate, Discount, Brokerage and Commission, from one (progressing $\frac{1}{8}$) to five per cent. on £1 (progressing £1) to £100, in consecutive order. The tables are well printed in clear figures, and the volume is of breast-coat pocket size.

Useful Engineers' Constants for the Slide-Rule, and how they are obtained. By J. A. Burns. Third edition. Pp. 68. 2s. net. 1921. (Percival Marshall.)

This little handbook gives instructions for using the slide-rule, with easy examples, and then by means of more difficult examples it shows the young engineer the best way to deal with the appropriate formulae for the problems that meet him in the every-day life of the works. As a rule the exposition is quite clear; the following is an exception:

"To what extent must a number be increased so as to leave the number when a given percentage is deducted from the new quantity?"

Examples in Differential and Integral Calculus. With Answers. By the late C. S. Jackson. Pp. 142. 10s. 6d. net. 1921. (Longmans, Green & Co.)

Mr. Jackson's examples and problems are to a very large extent of his own construction. His long experience at "The Shop," and as an examiner for public bodies, made him, perforce, an accomplished craftsman in the manufacture of problems exhibiting simple applications of mathematical principles to practical affairs of every-day life. As Mr. W. M. Roberts says in the Preface, they should prove particularly useful to teachers who are on the look-out for exercises which are not mere algebraical manipulations. The volume is one of those published in the well-known Modern Mathematical Series of Messrs. Longmans.

Simplified Arithmetic. Twentieth Century Methods for Home Students and Busy Workers. By W. H. F. Murdoch. Pp. 52. 2s. net. 1922. (Bowman and Murdoch.)

This is a little handbook for the private student, being an "attempt to give the essentials of arithmetic as concisely as possible." It opens with instructions for performing multiplication and division, followed by illustrations of the use of Napier's Bones. The hints for short methods of multiplication by 50 and 25 might have been enriched by similar hints for 5 and 125; similar hints for division would not be out of place. Applications follow of the formulae $(1+x)(1+y)$, $(1+x)/(1-x)$, where x and y are very small, etc. Rules for criteria of divisibility are given only for 2, 3, 4, and 5. A Prime number is defined as one "not divisible by any other number except itself." Simple explanation of Stocks and shares, Proportion, etc., each followed by four or five questions bring us to Fractions, Square roots and Interest, Simple and Compound, with tables both for compound interest and for present values and freeholds, leases, or annuities. An explanation of duodecimals is followed by a few such questions as: find the value of $3' : 6'' \times 9' : 2'' \times 11' \times 7'$. After two and a half pages on simple equations of one and of two unknowns, we are led through indices to logarithms, and the slide-rule is simply explained. "It is needless to remind readers that Napier . . . invented logarithms, and consequently the slide-rule." The final pages are given up to weights and measures. The idea of the booklet is sound, but we fear that the scale on which it is conceived is too small, except perhaps to those who merely require their Arithmetic to be "brushed up."

Commercial Arithmetic. By W. G. Borchardt. Pp. viii+200+24. 3s. 6d. 1922. (Rivington.)

Finance. Vol. I. **Banking: Stockbroking: Currency: Exchange.** By A. Risdon Palmer. Pp. xiii+189. Vol. II. **Finance.** Pp. xi+128. Vol. I. 5s. net. Vol. II. 4s. net. 1922. (Bell & Sons.)

Mr. Borchardt's volume is framed on the recommendations of the B. of E. Circular, 1116, on the First Year's Course of Instruction in Evening Schools. For the sake of those not familiar with the ground covered in this course, it is well to say that the first four rules, and vulgar and decimal fractions, are revised; the rectangle, parallelogram, triangle, trapezium, circle and cylinder are studied; simple book-keeping, percentages, discount, profit and loss, com-

mission, brokerage, insurance, cheques, etc., are dealt with in turn. Stress is laid throughout on short methods, rough checks and general accuracy. There is an excellent collection of concrete examples, the ground is well covered, the exposition is what we expect from a teacher of Mr. Borchardt's experience.

Mr. Risdon Palmer's two small volumes are for another class of student, and form an extremely successful modernisation of the subject for examinees, for people in business, for the citizen, and for the foreign student studying British methods. An enthusiast handed the books back to the writer, with the probably eulogistic but somewhat cryptic remark—"It's IT"! A close examination of *Finance* leaves us with what is no doubt the same impression. The work is thoroughly practical, the author has brought great expository power into play in his treatment of the more difficult parts, the examples are of a searching character, the documents given are from real life, and the author has availed himself of the experience of experts in every branch of the subject. We cannot speak too highly of this addition to Bell's *Handbooks of Commerce and Finance.*

An Experiment in Number Teaching. By J. B. Thomson (Mrs. Davies). Pp. 86. 2s. 6d. 1922. (Longmans, Green.)

Mrs. Davies, the Lecturer in Mathematics at the Mather Training College, Manchester, gives in these pages a "little record" of a most important experiment made at the Crimworth Demonstration School, attached to that institution. With the co-operation of colleagues more familiar than herself with the minds of young children, groping their way amid the mysteries of number, she describes the various devices contrived to meet the usual difficulties of children up to seven or eight years of age.

The idea of a syllabus is at present swept on one side. "The centre of gravity of the mathematical education of our younger pupils is now shifting from the curriculum to the child. The aim is first to discover the natural sequence of the child's development." And the author claims that the results of the search are already satisfactory both for teacher and pupil. Most interesting is the account of the ingenious ways in which a suitable environment was provided enabling the young theorist to become a practical person. No teacher can be the worse for studying the short final chapter on "guiding principles."

The Rhythm of Education. An Address delivered to the Training College Association. By A. N. Whitehead. Pp. 30. 1s. net. 1922. (Christophers.)

The lecturer discusses a well-known principle, "that different subjects and modes of study shall be undertaken by pupils at fitting times when they have reached the proper stage of mental development." He attacks the principle that the easier subject should precede the harder. The infant must begin by learning to speak—no easy task, and it proceeds to learn to write—to correlate sounds with shapes. The "hardest task in mathematics is the study of the elements of algebra, and yet this stage must precede the comparative simplicity of the differential calculus. There are three stages in intellectual progress, those of romance, precision and generalisation. Without the facts of the stage of romance, vaguely apprehended in their broad generality, a further stage is barren." The facts of romance have disclosed ideas with possibilities of wide significance, and in the stage of precision we acquire other facts in a systematic order, which thereby form both a disclosure and an analysis of the general subject-matter of the romance. "When the necessary technique is acquired and the ideas are classified, we return in the third stage to romanticism." Education is the continued repetition of such cycles. The generalising spirit must dominate the University. "At school the boy painfully rises from the particular towards glimpses at general ideas; at the University he should start from general ideas and study their application to concrete cases . . . concrete fact should be studied as illustrating the scope of general ideas." Few teachers will fail to read without advantage this masterly and inspiring address.

Life and the Laws of Thermodynamics. The Twenty-fourth Boyle Lecture. Pp. 12. By SIR W. M. BAYLISS. 1s. net. 1922. (Oxford Univ. Press.)

This is a reprint of a lecture delivered before the Junior Scientific Club of the University Club of Oxford. The speaker gives illustrations of the importance of the laws of energetics in vital processes. For instance, the first stage of enzyme action is a case of adsorption, living beings transform, as a Daniell battery or an electric motor does, one form of energy into another, without loss of heat.

Papers set in the Mathematical Tripos. Part I. in the University of Cambridge, 1918-1922. Pp. 77. 3s. net. 1922. (Cambridge Univ. Press.)

Examples in Optics. Compiled by T. J. I'A. BROMWICH. Pp. 16. 2s. net. 1921. (Bowes & Bowes.)

No comment is necessary on the first of these booklets, which gives in a nicely printed and easily accessible form the Tripos (Part I.) papers for the last five years. Mr. Bromwich tells us that his examples are his old class-room collection at St. John's, which are now printed with additions in the present form to be more readily available for the work of Intercollegiate Lectures.

THE LIBRARY.

THE Library has now been removed to 29 Gordon Square, London, W.C. 1, and Mr. W. E. Paterson has taken over the duties of Honorary Librarian.

The Librarian will gladly receive and acknowledge in the *Gazette* any donation of ancient or modern works on mathematical subjects.

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Robert Recorde. By F. CAJORI. Pp. 294-302. (Reprint from *The Mathematics Teacher.* May, 1922.)

First Ideas of Geometry. By G. ST. L. CARSON and D. E. SMITH. (Part I. Section I. of the Authors' *Plane Geometry.*) Pp. 90. 1s. 6d. net. 1922. (Ginn.)

Vector Analysis and the Theory of Relativity. By F. D. MURNAGHAN. Pp. x + 125. n.p. 1922. (Johns Hopkins Press, Baltimore, Md.)

Analytische Behandeling van de rationale Kromme van den vierden Graad in een Vierdimensionale Ruimte. By J. FR. DE VRIES. Pp. xi, 159. 4 gld. 1922. (Nijhoff, 's-Gravenhage.)

Conférences sur les Transformations en Géométrie Plane. By W. DE TANNENBERG. Pp. 50. 4 fr. 1922. (Vuibert.)

Calcul des Erreurs absolues et des Erreurs relatives. By W. DE TANNENBERG. Pp. 35. n.p. 1922. (Vuibert.)

La Composition de Mathématiques dans l'Examen d'Admission à l'École Polytechnique de 1901 *à* 1921. By F. MICHEL and M. POTRON. Exercices d'Application du Cours de Mathématiques Spéciales. Pp. xii + 452. n.p. (Gauthier-Villars.)

Cours Complet de Mathématiques Spéciales. Tome III. *Mécanique.* By J. HAAG. Pp. viii + 192. 12 fr. 1922. (Gauthier-Villars.)

A Syllabus on American History and the Natural Sciences. By A. B. HULBERT. Pp. 389-396. Reprint from *Colorado College Publications.*

On the Faraday-Tube Theory of Electro-Magnetism. By W. G. BROWN. Pp. 225-246. Reprint from *Proc. Roy. Soc. Edin.*, vol. 42, part ii. No. 16.

A Treatise on the Integral Calculus, with Applications, Examples and Problems. Vol. II. By J. EDWARDS. Pp. xv + 980. 50s. 1922. (Macmillan.)

The Principle of Relativity, with Applications to Physical Science. By A. N. WHITEHEAD. Pp. xii + 190. 10s. 6d. net. 1922. (Cambridge University Press.)

Can Waste of Mental Energy be avoided? By F. CAJORI. Pp. 355-356. Reprint from *Science*, Sept. 29, 1922.

Operationes super Magnitudines. By G. PEANO. Pp. 269-283. Reprint from *Rassegna di Mat. e Fis.* ii. 12. (Ferraris, Rome.)

Mathematical Tables. By G. H. BRYAN. Pp. 28. 3s. 6d. 1922. (Macmillan.)

Disagreeing with the Text-book. Pp. 449-454. By G. A. MILLER. Reprint from *School and Society.* Oct. 21, 1922.

Cours Complet de Mathématiques Spéciales. Tome III. *Mécanique.* By J. HAAG. Pp. viii + 189. 16 fr. 1922. (Gauthier-Villars.)

La Composition de Mathématiques dans l'examen d'admission à l'École Polytechnique de 1901-1920. By F. MICHEL et M. POTRON. Pp. xiii + 452. 40 fr. 1922. (Gauthier-Villars.)

Introduction à la Théorie de la Relativité : Calcul Différentiel absolu et Géométrie. By H. GALBRUN. Pp. x + 460. 60 fr. 1922. (Gauthier-Villars.)

The Quantum Theory. By F. REICHE. Translated by H. S. HATFIELD and H. L. BROSE. Pp. 183. 6s. net. 1922. (Methuen.)

Life Contingencies. By E. F. SPURGEON. Pp. xxvi + 477. 30s. net. 1922. (C. & E. Layton.)

Calculus and Probability for Actuarial Students. By A. HENRY. Pp. 152. 12s. 6d. net. 1922. (C. & E. Layton.)

A Treatise on the Theory of Bessel Functions. By G. N. WATSON. Pp. viii + 804. £3 10s. net. 1922. (Cam. Univ. Press.)

Dimensional Analysis. By P. W. BRIDGMAN. Pp. 112. 25s. net. 1922. (Yale Univ. Press : Oxford Univ. Press.)

Mathematics and Physical Science in Classical Antiquity. Translated from the German of J. L. HEIBERG by D. C. MACGREGOR. Pp. 110. 2s. 6d. net. 1922. (Oxford University Press.)

Principles of Geometry. Vol. II. *Plane Geometry : Conics, Circles, Non-Euclidean-Geometry.* By H. F. BAKER. Pp. xv + 243. 15s. net. 1922. (Camb. Univ. Press.)

A New Manual of Logarithms to Seven Places of Decimals. Edited by DR. BRUHNS. 13th Stereotyped Edition. Pp. xxiv + 610. 15s. net. 1922. (Chapman & Hall.)

Elementi di Aritmetica, con note storiche e numerose questioni varie per le Scuole Medie Superiore. 6th Edition. By GAETANO FAZZARI. Pp. 196. 7 l. 1922. (Trimarchi, Palermo.)

The Theory of Spectra and Atomic Constitution. Three Essays by NIELS BOHR. Pp. vii + 126. 7s. 6d. net. 1922. (Camb. Univ. Press.)

Vector Calculus, with Applications to Physics. By J. B. SHAW. Pp. v + 314. 14s. net. (Constable.)

Haupt-Exponents, Residue-Indices, and Standard Congruences. By Lt.-Col. ALLAN CUNNINGHAM, R.E., H. J. WOODALL, and (the late) T. G. CREAK. Pp. viii + 136. n.p. 1922. (F. Hodgson.)

Prolegomena to Analytical Geometry in Anisotropic Euclidean Space of Three Dimensions. By E. H. NEVILLE. Pp. xxii + 368. 30s. net. 1922. (Camb. Univ. Press.)

American Journal of Mathematics. (Johns Hopkins Press, Baltimore.)

Jan. 1921.

An Arithmetical Dual of Kummer's Quartic Surface. Pp. 1-11. E. T. BELL. *Incidences of Straight Lines and Plane Algebraic Curves and Surfaces generated by them.* Pp. 12-19. A. EMCH. *On the Theorems of Gauss and Green.* Pp. 20-24. V. C. POOR. *An Extension of the Sturm-Liouville Expansion.* Pp. 25-53. C. C. CAMP. *Conformal Transformations of Period n and Groups generated by them.* Pp. 54-86. H. LANGMAN.

The American Mathematical Monthly. (Lancaster, Pa.)

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Serret's Analogue of Desargues's Theorem. Pp. 111-112. H. S. WHITE. *A certain Two-dimensional Locus.* [The loci of points z_1, z_2 are respectively the interiors (boundaries included) of two circles c_1 and c_2. Find the locus of a point z equidistant from z_1 and z_2.] Pp. 112-114. J. L. WALSH. *Among my Autographs.* [Babbage, De Morgan.] Pp. 114-116. D. E. SMITH. *A Method of deriving Formulae for the Expansion of* sin $(x+y)$, cos $(x+y)$. Pp. 120-121. H. W. WEBB. *The Derivation of Formulae in the Mathematics of Investment.* P. 122 C. N. REYNOLDS.

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The Imaginary Points of Geometry. Pp. 145-149. A. A. BENNETT. *Lines of Illumination caused by the Passage of Light through Screens.* Pp. 149-156. W. H. ROEVER. *Among my Autographs* [Brewster, Clifford]. Pp. 157-158. D. E. SMITH. *On Proofs by Mathematical Induction.* Pp. 162-164. R. S. HOAR : A. A. BENNETT.

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Spanish and Portuguese Symbols for "Thousands." Pp. 201-202. F. CAJORI. *A General Construction of Circular Cubics.* Pp. 202-204. R. M. MATHEWS. *A Generalisation of the Strophoid.* Pp. 204-207. J. H. WEAVER. *Among my Autographs* [Sir W. Rowan Hamilton (1847)]. Pp. 209-210. D. E. SMITH. *Solutions of a Problem in Skeleton Division.* Pp. 211-213. D. R. CURTISS, A. A. BENNETT. *What is a Calculus?* Pp. 213-217. J. P. BALLANTINE and the EDITOR.

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